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Topics in Algorithmic Game Theory and Economics, Exercise Sheet 3

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter20/game-theory

Total Points: 4 + 40 = 44 Due: Thursday, **Jan. 21, 23:59 (CET)**, 2020

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, using your own words. ***Always explain your answers.*** *Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of the total points on exercise sheets to be admitted to the exam. Send your solutions in* ***PDF*** *format directly to Golnoosh (**gshahkar@mpi-inf.mpg.de**). You get the first 4 points if you hand in typed solutions.*

Exercise 1 — 10 points

Consider the following algorithm $\mathcal{A}$ for the secretary problem (assume that m is even).

For ordering $\sigma = (\sigma(1), \dots, \sigma(m))$, do the following.
Phase I (Observation):
- For $i = 1, \dots, \frac{m}{2}$: Do not select $\sigma(i)$.

Phase II (Selection):
- Set threshold $t = \max_{j=1,\dots,\frac{m}{2}} w_{\sigma(j)}$.
- For $i = \frac{m}{2} + 1, \dots, m$: If $w_{\sigma(i)} \geq t$, select $\sigma(i)$ and STOP.

Show that this algorithm gives a $\frac{1}{4}$-approximation for the secretary problem (with uniform random arrival order). That is,

$$\mathbb{E}_\sigma(w(\mathcal{A}(\sigma))) \geq \frac{1}{4} \max_i w_i,$$

where $w(\mathcal{A}(\sigma))$ is the weight of the element selected by the algorithm.

Exercise 2 — 5+5 points

Consider the setting where we want to select one element from $\{e_1, \dots, e_m\}$ online (with unknown weights $w_1, \dots, w_m$), under a **worst-case arrival order** σ.

(a) Show that there is an online (deterministic or randomized) algorithm $\mathcal{A}$, for which

$$\min_\sigma w(\mathcal{A}(\sigma)) \geq \frac{1}{m} w^*.$$

where $w^* = \max_e w_e$ and $w(\mathcal{A}(\sigma))$ the (expected) weight of the element selected by $\mathcal{A}$. *(That is, in case $\mathcal{A}$ is randomized, $w(\mathcal{A}(\sigma))$ denotes the expected weight of the selected element. The expectation here is taken w.r.t. the random choices of the algorithm.)*

(b) Next, consider the case $m = 2$. Show that there is no (deterministic or randomized) online algorithm so that for every choice of weights w_1 and w_2

$$\min_\sigma w(\mathcal{A}(\sigma)) \geq \alpha \cdot w^*.$$

for any constant $\alpha > \frac{1}{2}$.

Exercise 3 — 10 points

Consider the following (offline) auction setting. We have n bidders and items $\{1,\ldots,k\}$ with $k < n$. Each item can be assigned to at most one bidder. Each bidder has a private valuation v_i for receiving one of the items. (This means the items are identical, i.e., for every bidder i, $v_{i\ell} = v_{i\ell'}$ for every two items ℓ and ℓ'.) Bidder i declares a single bid $b_i \geq 0$ for receiving one of the k (identical) items.

Give a deterministic, strategy proof (meaning bidding truthfully is optimal), individually rational, welfare optimizing mechanism (x,p) that runs in time $O(n\log(n))$. The input of the mechanism is the vector $b = (b_1,\ldots,b_n)$. The utilities of the players are given by

$$u_i(b) = \begin{cases} v_i - p_j(b) & \text{if i receives item } j, \\ 0 & \text{if i does not get an item,} \end{cases}$$

for bid vector $b = (b_1,\ldots,b_n)$.

Exercise 4 — 10 points

Consider the setting of selling one item online under a uniform random arrival order of bidders in $\{1,\ldots,n\}$, where bidder i has (true) valuation $v_i \geq 0$ for the item. Let $v^* = \max_i v_i$.

Give a strategyproof online mechanism $\mathcal{M} = (x,p)$, for which

$$\mathbb{E}_\sigma[v(\mathcal{M}(\sigma))] \geq \left(\frac{1}{e} - \frac{1}{n}\right) \cdot v^*.$$

where σ is the arrival order and $v(\mathcal{M}(\sigma))$ the value of the bidder that receives the item (if any).