Topics in Algorithmic Game Theory and Economics

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Max Planck Institute for Informatics
Saarland Informatics Campus

January 6, 2020

Lecture 7
Online Selection Problems

(Offline) selection problems

Selection problem

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- Spanning trees in given graph;
- Bases of a matroid;
- Matchings in given (bipartite) graph.

Some examples

Maximum weight spanning tree

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Example (Spanning tree)

Given is undirected graph $G = (V, E)$ with edge-weight function $w : E \rightarrow \mathbb{R}_{\geq 0}$.

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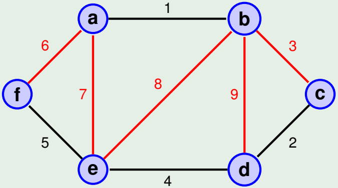

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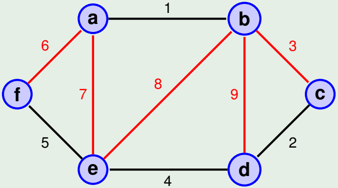

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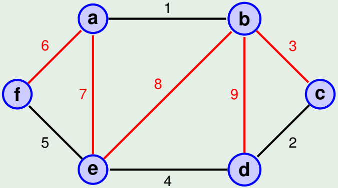

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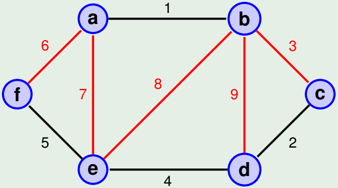

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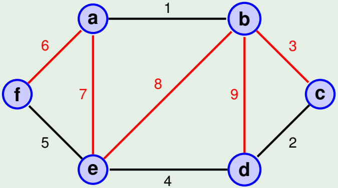

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- *Downward-closed*: $A \in \mathcal{F}$ and $B \subseteq A \Rightarrow B \in \mathcal{F}$,
- *Augmentation property*:

  $A, C \in \mathcal{F}$ and $|C| > |A| \Rightarrow \exists e \in C \setminus A$ such that $A \cup \{e\} \in \mathcal{F}$.

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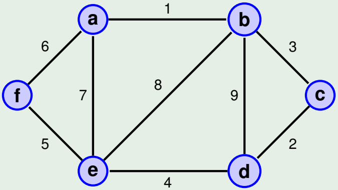

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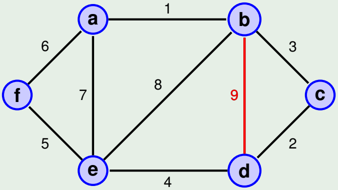

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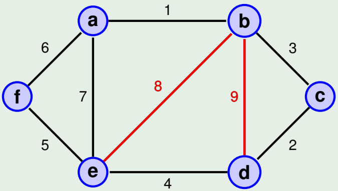

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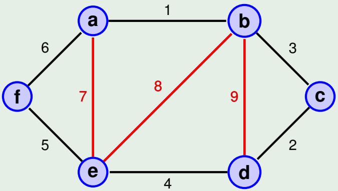

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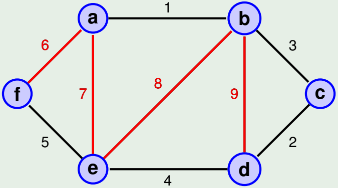

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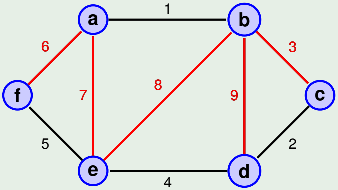

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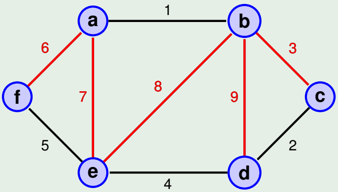

As the weights are non-negative, the output of the greedy algorithm is always a maximum weight spanning tree.

Maximum weight bipartite matching

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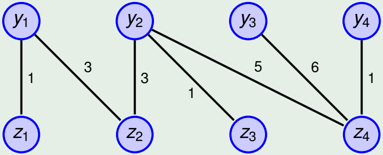

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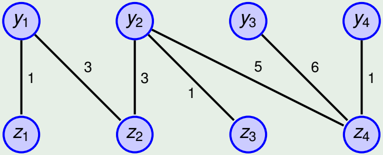

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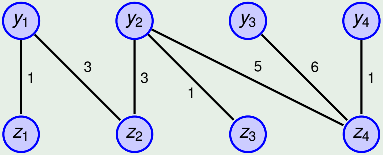

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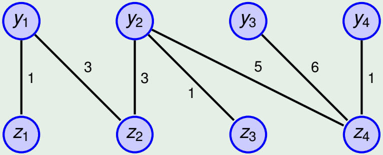

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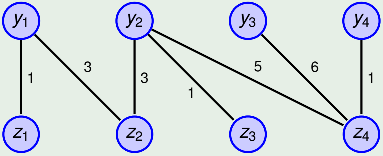

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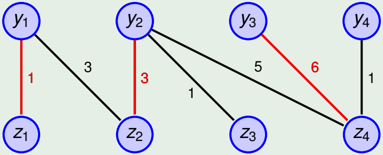

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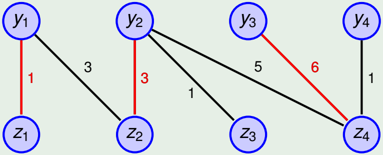

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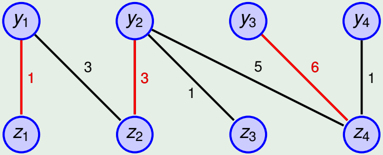

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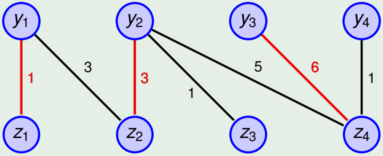

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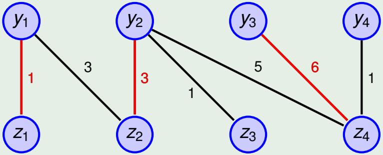

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- Well-known combinatorial algorithm: Hungarian method.

Approximation algorithms

Sometimes solving the program

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> **Goal:** Compute (in poly-time) $X \in \mathcal{F}$ such that
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> $$w(X) := \sum_{e \in X} w(e) \geq \alpha \cdot \mathrm{OPT}.$$

Computational aspects

Given is

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- Computational complexity measured in terms of m, representation size of weights $w(e)$, and number of oracle calls.

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Elements arrive one by one in unknown order $\sigma = (\sigma(1), \ldots, \sigma(m))$.

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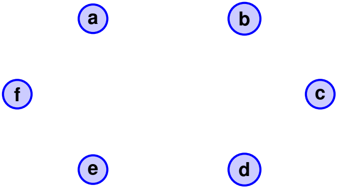

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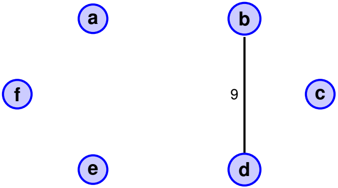

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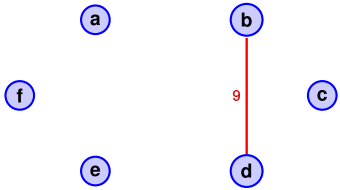

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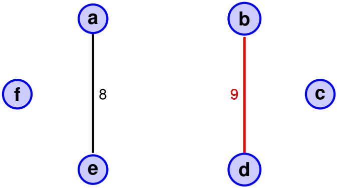

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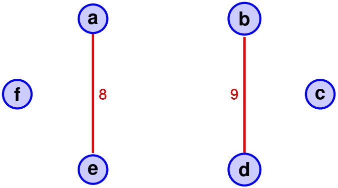

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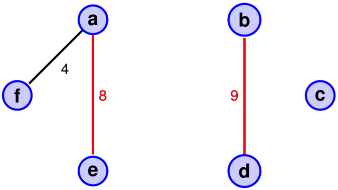

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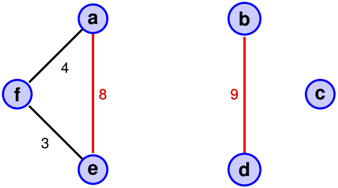

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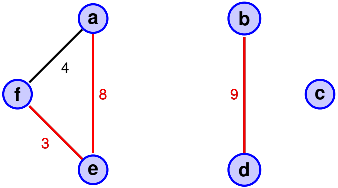

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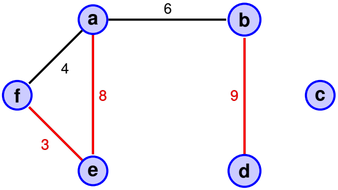

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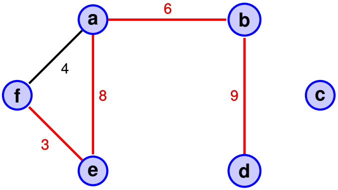

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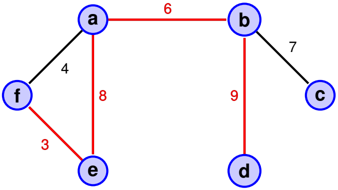

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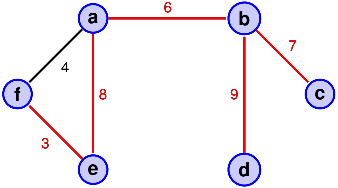

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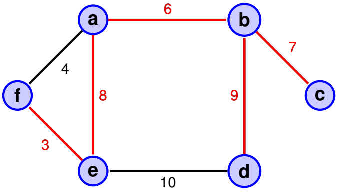

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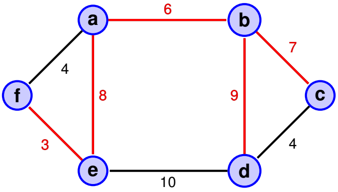

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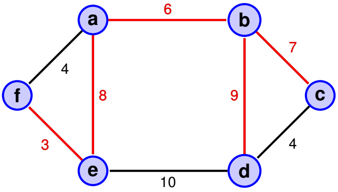

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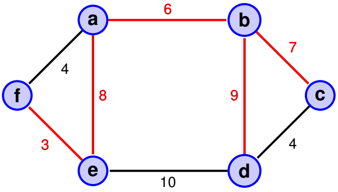

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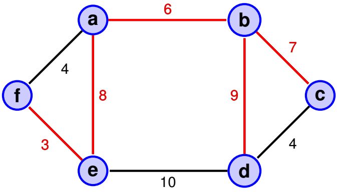

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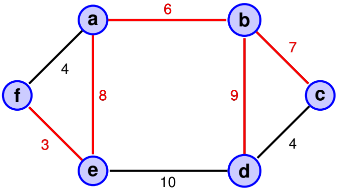

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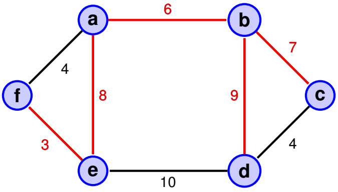

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Online selection problems

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- In worst-case arrival setting, no constant-factor algorithm exists.

Online selection problems

Uniform random arrivals: Secretary problem

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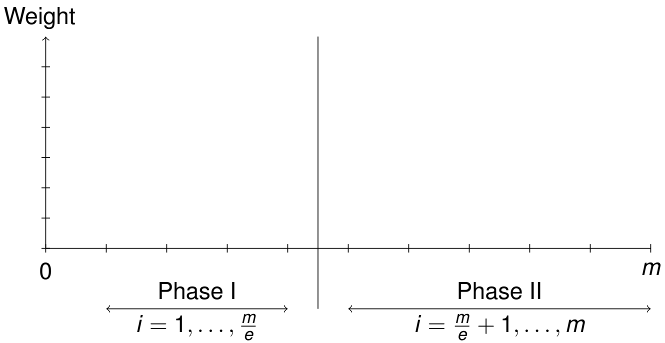

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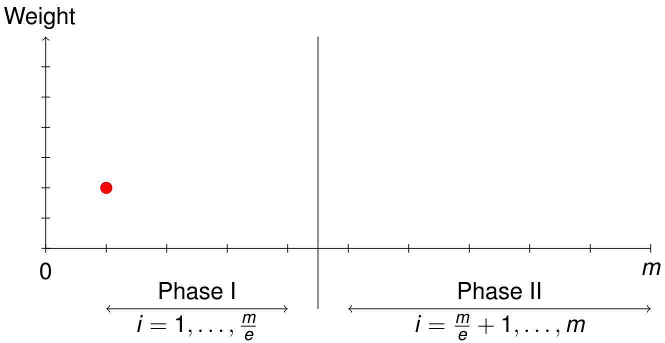

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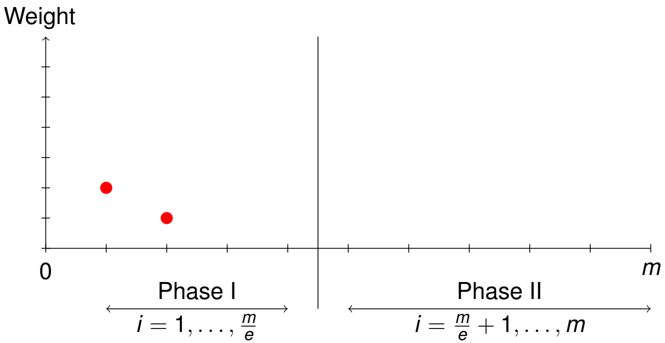

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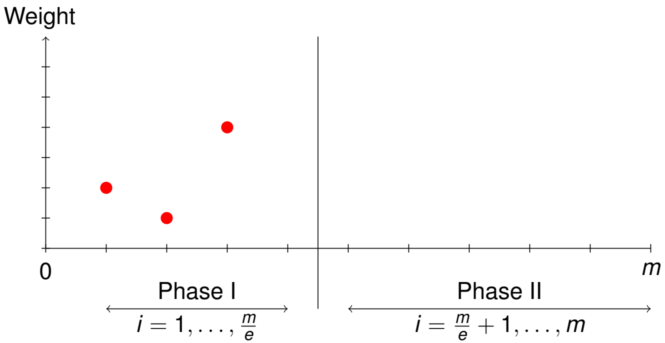

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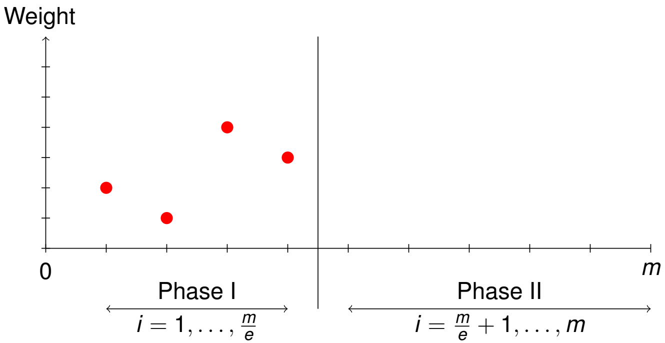

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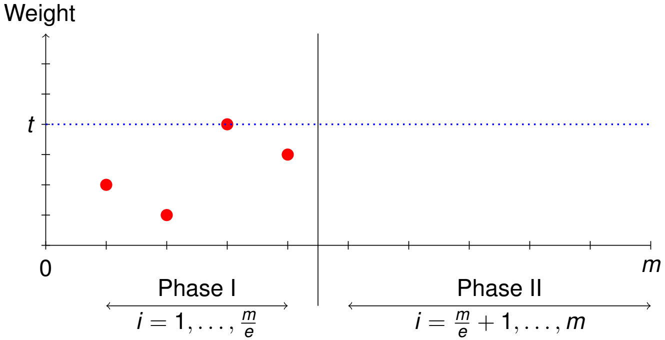

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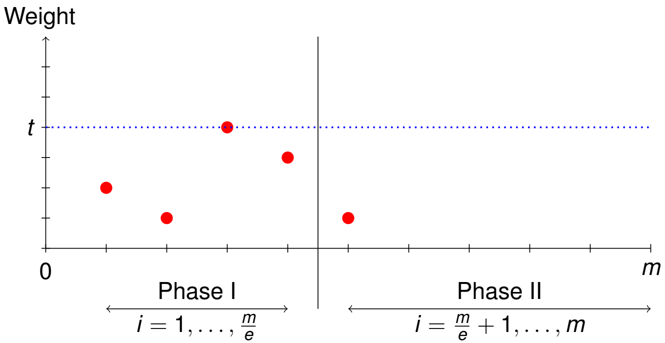

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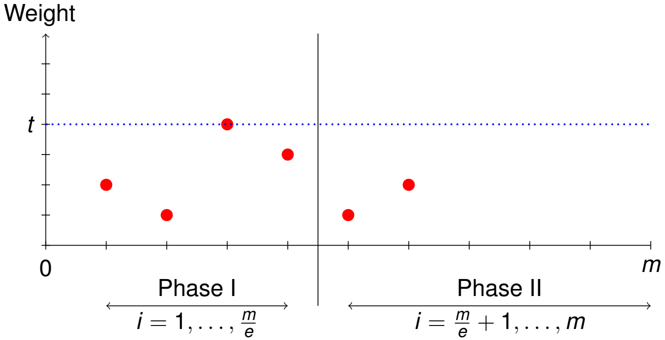

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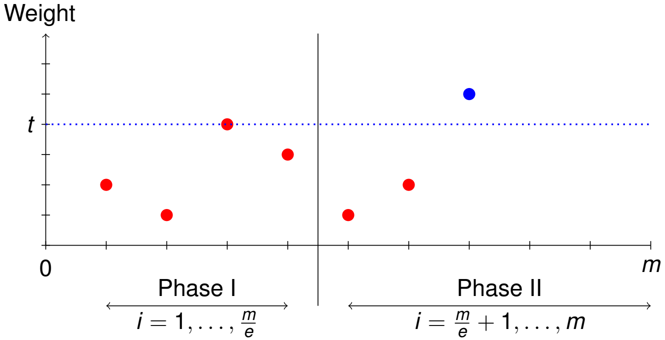

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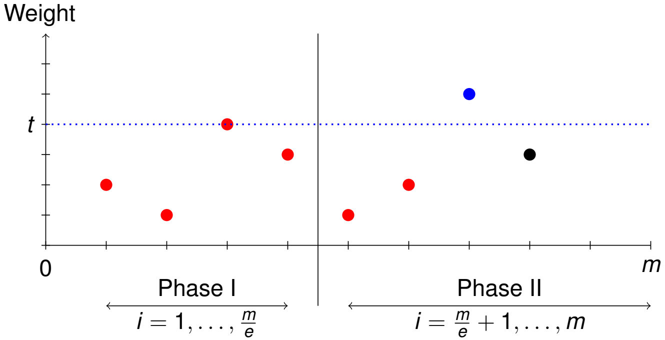

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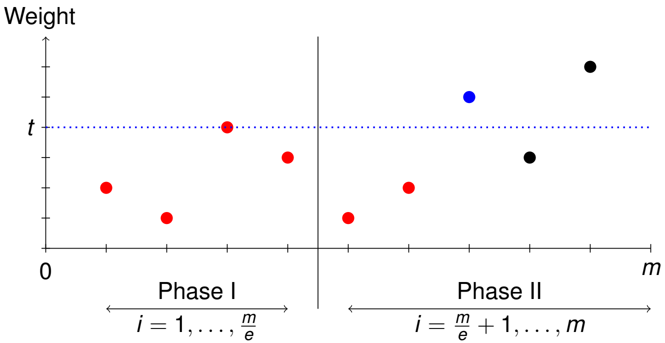

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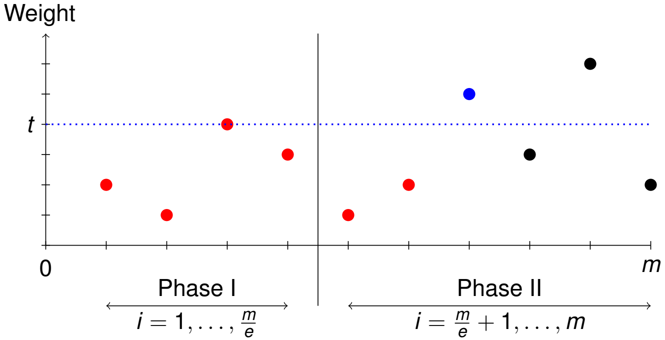

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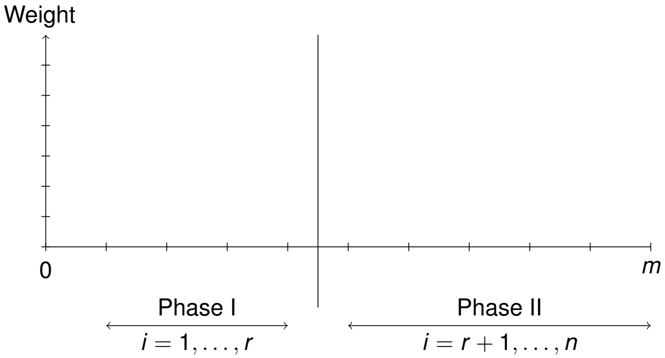

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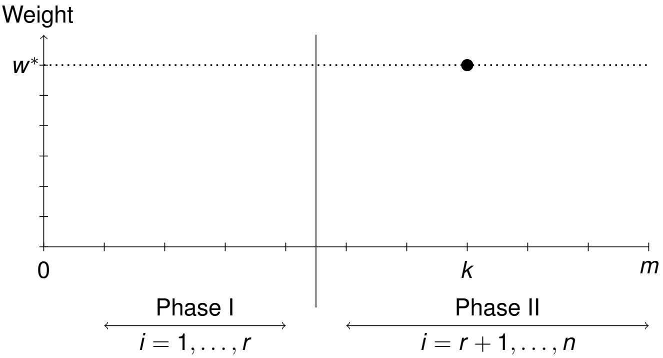

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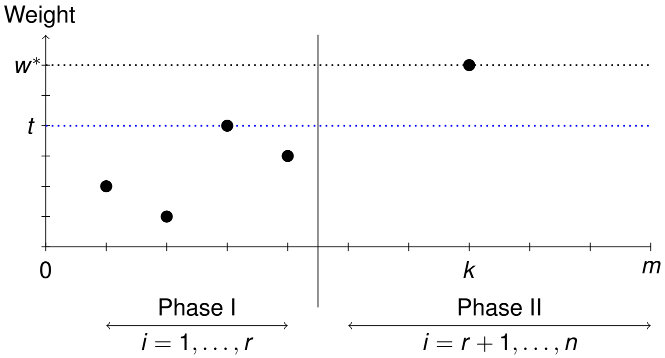

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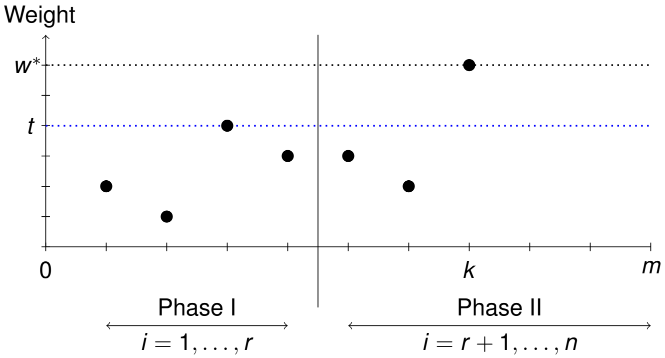

Analysis (sketch)

Consider $w^* = \max_i w_i$ (assume w.l.o.g. that weights are distinct).

- For readability, we assume (wrongfully) that $r = m/e$ is integer.
  - Simple adjustments to analysis suffice to deal with this issue.

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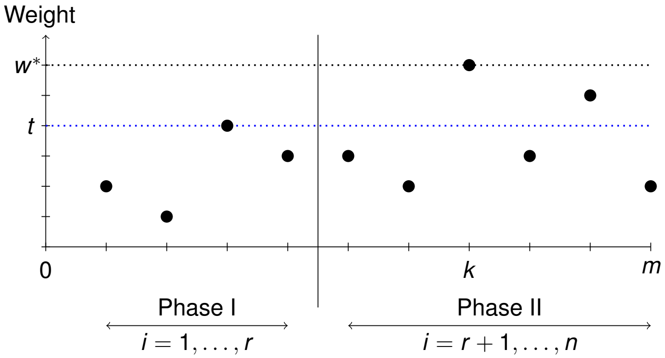

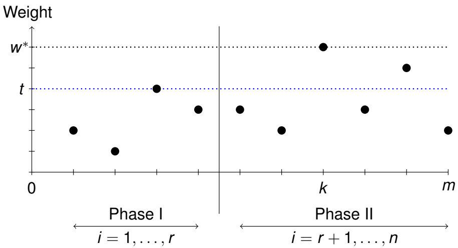

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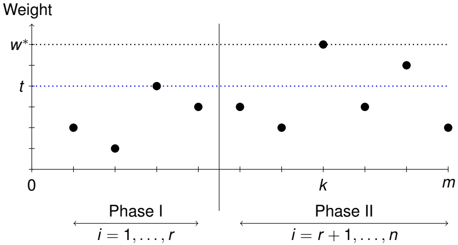

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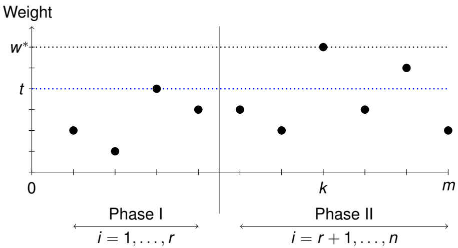

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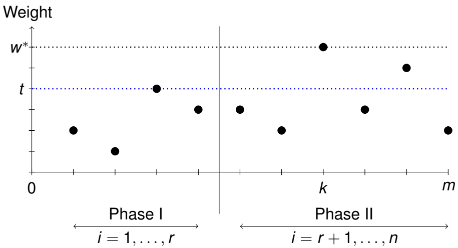

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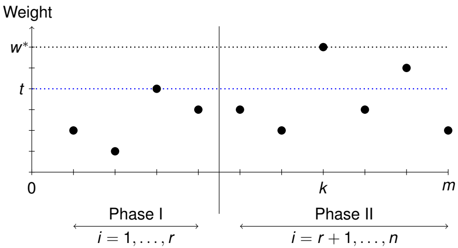

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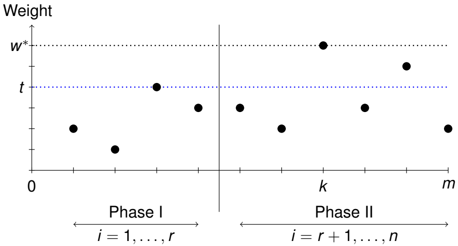

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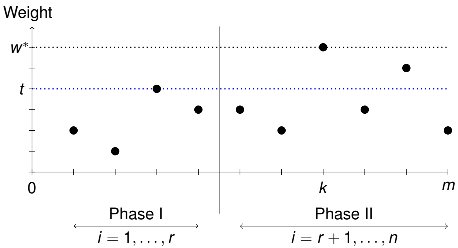

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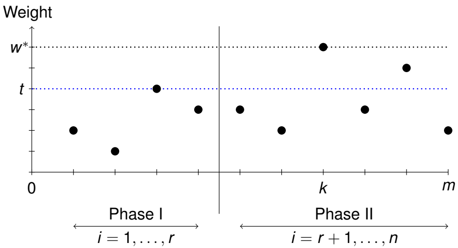

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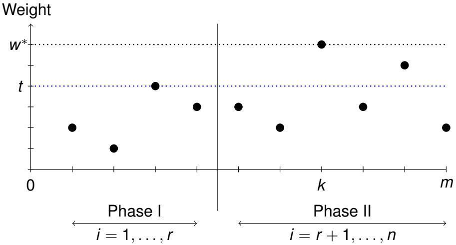

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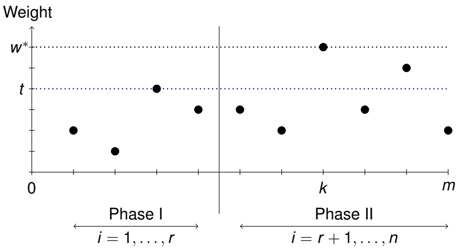

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Online selection problems

Prophet Inequalities

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Adversary is non-adaptive if order is fixed after seeing all realizations.

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- Performance objective is formalized next.

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Then

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- I.e., optimal algorithm only half as bad as prophet ($\alpha = \frac{1}{2}$).

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- Will see some (offline) mechanism design basics next week.