Problem 1 (Matricization). Let $\mathcal{T} \in \mathbb{R}^{5\times3\times3}$ be a tensor whose frontal slices are

$$\boldsymbol{T}_1 = \begin{pmatrix} 1 & 11 & 29 \\ 2 & 13 & 31 \\ 3 & 17 & 37 \\ 5 & 19 & 41 \\ 7 & 23 & 43 \end{pmatrix}, \quad \boldsymbol{T}_2 = \begin{pmatrix} 47 & 71 & 97 \\ 53 & 73 & 101 \\ 59 & 79 & 103 \\ 61 & 83 & 107 \\ 67 & 89 & 109 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{T}_3 = \begin{pmatrix} 113 & 149 & 173 \\ 127 & 151 & 179 \\ 131 & 157 & 181 \\ 137 & 163 & 191 \\ 139 & 167 & 193 \end{pmatrix}. \tag{1.1}$$

Write open

a) the matrix $\boldsymbol{T}_{(1)}$

b) the first five columns and the total dimensions of $\boldsymbol{T}_{(2)}$

c) the first five columns and the total dimensions of $\boldsymbol{T}_{(3)}$

Solution.

a)

$$\boldsymbol{T}_{(1)} = \begin{pmatrix} 1 & 11 & 29 & 47 & 71 & 97 & 113 & 149 & 173 \\ 2 & 13 & 31 & 53 & 73 & 101 & 127 & 151 & 179 \\ 3 & 17 & 37 & 59 & 79 & 103 & 131 & 157 & 181 \\ 5 & 19 & 41 & 61 & 83 & 107 & 137 & 163 & 191 \\ 7 & 23 & 43 & 67 & 89 & 109 & 139 & 167 & 193 \end{pmatrix}$$

b)

$$\boldsymbol{T}_{(2)}(:, 1:5) = \begin{pmatrix} 1 & 2 & 3 & 5 & 7 \\ 11 & 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 & 43 \end{pmatrix} \quad \boldsymbol{T}_{(2)} \in \mathbb{R}^{3\times15}$$

c)

$$\boldsymbol{T}_{(3)}(:, 1:5) = \begin{pmatrix} 1 & 2 & 3 & 5 & 7 \\ 47 & 53 & 59 & 61 & 67 \\ 113 & 127 & 131 & 137 & 139 \end{pmatrix} \quad \boldsymbol{T}_{(3)} \in \mathbb{R}^{3\times15}$$

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Problem 2 (Tensor-vector product). Let $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{5\times3\times3}$ be as in (1.1) and let

$$\boldsymbol{v} = \begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{pmatrix}1\\2\\4\end{pmatrix}.$$

Calculate

a) $\boldsymbol{\mathcal{T}} \,\bar{\times}_1\, \boldsymbol{v}$

b) $\boldsymbol{\mathcal{T}} \,\bar{\times}_3\, \boldsymbol{w}$.

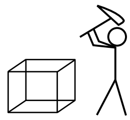
Solution.

a)

$$\boldsymbol{\mathcal{T}} \,\bar{\times}_1\, \boldsymbol{v} = \begin{pmatrix}69 & 909 & 2003\\279 & 1231 & 2409\\581 & 1581 & 2803\end{pmatrix}$$

b)

$$\boldsymbol{\mathcal{T}} \,\bar{\times}_3\, \boldsymbol{w} = \begin{pmatrix}547 & 749 & 915\\616 & 763 & 949\\645 & 803 & 967\\675 & 837 & 1019\\697 & 869 & 1033\end{pmatrix}$$



Problem 3 (Tensor-matrix product). Let $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{5\times3\times3}$ be as in (1.1) and let

$$\boldsymbol{M} = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix} .$$

Calculate $\boldsymbol{\mathcal{T}} \times_1 \boldsymbol{M}$.

Solution. The result is a 2-by-3-by-3 tensor with frontal slices as follows:

$$(\boldsymbol{\mathcal{T}} \times_1 \boldsymbol{M})_1 = \begin{pmatrix} 36 & 166 & 362 \\ 54 & 249 & 543 \end{pmatrix}$$
$$(\boldsymbol{\mathcal{T}} \times_1 \boldsymbol{M})_2 = \begin{pmatrix} 574 & 790 & 1034 \\ 861 & 1185 & 1551 \end{pmatrix}$$
$$(\boldsymbol{\mathcal{T}} \times_1 \boldsymbol{M})_3 = \begin{pmatrix} 1294 & 1574 & 1834 \\ 1941 & 2361 & 2751 \end{pmatrix}$$

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Problem 4 (Tensors as multilinear maps). Matrices are often presented as linear maps between two vector spaces, $\boldsymbol{M}\colon \mathbb{R}^m \to \mathbb{R}^n$ with $\boldsymbol{M}(\boldsymbol{v}) = \boldsymbol{M}\boldsymbol{v}$. In some literature (especially physics), tensors are considered as *multi-linear* maps from product vector spaces to real numbers. For example, if $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{I\times J\times K}$, we interpret $\boldsymbol{\mathcal{T}}\colon \mathbb{R}^I \times \mathbb{R}^J \times \mathbb{R}^K \to \mathbb{R}$. The multi-linear map is defined as

$$\boldsymbol{\mathcal{T}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{\mathcal{T}} \bar{\times}_1 \boldsymbol{u} \bar{\times}_1 \boldsymbol{v} \bar{\times}_1 \boldsymbol{w} \,.$$

Let $\boldsymbol{\mathcal{T}}$ be as in (1.1) and let

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{w} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}.$$

Calculate $\boldsymbol{\mathcal{T}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$.

N.B. The original version of this question had a typo in the first equation, which has been corrected.

Solution.

$$\boldsymbol{\mathcal{T}} \bar{\times}_1 \boldsymbol{u} = \begin{pmatrix} 69 & 909 & 2003 \\ 279 & 1231 & 2409 \\ 581 & 1581 & 2803 \end{pmatrix} \tag{4.1}$$

$$(\boldsymbol{\mathcal{T}} \bar{\times}_1 \boldsymbol{u}) \bar{\times}_1 \boldsymbol{v} = \begin{pmatrix} 2951 \\ 9695 \\ 18033 \end{pmatrix} \tag{4.2}$$

$$((\boldsymbol{\mathcal{T}} \bar{\times}_1 \boldsymbol{u}) \bar{\times}_1 \boldsymbol{v}) \bar{\times}_1 \boldsymbol{w} = 183559 \tag{4.3}$$

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