D5: Databases and Information Systems
Tensors in Data Analysis, WS 2017–18
Homework #2: CP decomposition and tensor rank
Tutorial: **11 October 2017** at 14:15

You can discuss these problems with other students, but everybody must do and present their own answers. You can use computers etc. to perform the algebraic operations, but you must show the intermediate steps (and "computer said so" is never a valid answer). You are of course free to use material from the Internet, but again, you must present the intermediate steps and you must also be able to explain why the steps are valid and why you chose them. You can mark an answer even if it is not complete or correct, as long as you have made significant progress towards solving it. Note, however, that the lecturer does the final decision on whether your solution is complete (or correct) enough for a mark.

Problem 1 (Khatri–Rao is associative). Let $\boldsymbol{A} \in \mathbb{R}^{I \times L}$, $\boldsymbol{B} \in \mathbb{R}^{J \times L}$, and $\boldsymbol{C} \in \mathbb{R}^{K \times L}$. Show that

$$(\boldsymbol{A} \odot \boldsymbol{B}) \odot \boldsymbol{C} = \boldsymbol{A} \odot (\boldsymbol{B} \odot \boldsymbol{C}) \ .$$

Problem 2 (Khatri–Rao and pseudo-inverse). Recall that the Moore–Penrose pseudo-inverse of a matrix $\boldsymbol{M}$ is a matrix $\boldsymbol{M}^+$ such that

$$\boldsymbol{M}\boldsymbol{M}^+\boldsymbol{M} = \boldsymbol{M} \tag{2.1}$$

$$\boldsymbol{M}^+\boldsymbol{M}\boldsymbol{M}^+ = \boldsymbol{M}^+ \tag{2.2}$$

$$(\boldsymbol{M}\boldsymbol{M}^+)^T = \boldsymbol{M}\boldsymbol{M}^+ \tag{2.3}$$

$$(\boldsymbol{M}^+\boldsymbol{M})^T = \boldsymbol{M}^+\boldsymbol{M} \ . \tag{2.4}$$

Let $\boldsymbol{A} \in \mathbb{R}^{I \times K}$ and $\boldsymbol{B} \in \mathbb{R}^{J \times K}$ be such that $K < \min\{I, J\}$ and $\operatorname{rank}\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big) = K$. Show that

$$(\boldsymbol{A} \odot \boldsymbol{B})^+ = \big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+ (\boldsymbol{A} \odot \boldsymbol{B})^T \ . \tag{2.5}$$

Hint: You can use the equation

$$(\boldsymbol{A} \odot \boldsymbol{B})^T (\boldsymbol{A} \odot \boldsymbol{B}) = \boldsymbol{A}^T\boldsymbol{A} * \boldsymbol{B}^T\boldsymbol{B} \ . \tag{2.6}$$

Problem 3 (CP decomposition). Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{C} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \ .$$

a) Calculate $\boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1$.

b) Calculate $\boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2$.

c) Calculate $[\![\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]\!]$.

Problem 4 (Uniqueness of a rank decomposition). In the lectures we saw a tensor $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{2 \times 2 \times 2}$,

$$\boldsymbol{T}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{T}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ,$$

that has real tensor rank 3. One factorization that obtains this rank 3 is

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{C} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \ .$$

What can you say about the uniqueness of this factorization?