

Problem 1 (Khatri–Rao is associative). Let $\mathbf{A} \in \mathbb{R}^{I \times L}$, $\mathbf{B} \in \mathbb{R}^{J \times L}$, and $\mathbf{C} \in \mathbb{R}^{K \times L}$. Show that

$$(\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C} = \mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}).$$

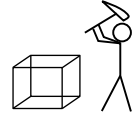
Solution. Let $\mathbf{D} = (\mathbf{A} \odot \mathbf{B})$ so that $d_{(i,j)\ell} = a_{i\ell}b_{j\ell}$. Then

$$\mathbf{D} \odot \mathbf{C} = [\mathbf{d}_1 \otimes \mathbf{c}_1 \quad \mathbf{d}_2 \otimes \mathbf{c}_2 \quad \dots \quad \mathbf{d}_L \otimes \mathbf{c}_L]$$

and

$$(\mathbf{D} \odot \mathbf{C})_{(i,j,k)\ell} = d_{(i,j)\ell}c_{k\ell} = a_{i\ell}b_{j\ell}c_{k\ell} = a_{i\ell}(\mathbf{b}_\ell \otimes \mathbf{c}_\ell)_{(j,k)\ell} = (\mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}))_{(i,j,k)\ell},$$

proving the claim.



Problem 2 (Khatri–Rao and pseudo-inverse). Recall that the Moore–Penrose pseudo-inverse of a matrix M is a matrix M^+ such that

$$MM^+M = M \quad (2.1)$$

$$M^+MM^+ = M^+ \quad (2.2)$$

$$(MM^+)^T = MM^+ \quad (2.3)$$

$$(M^+M)^T = M^+M. \quad (2.4)$$

Let $A \in \mathbb{R}^{I \times K}$ and $B \in \mathbb{R}^{J \times K}$ be such that $K < \min\{I, J\}$ and $\text{rank}((A^T A) * (B^T B)) = K$. Show that

$$(A \odot B)^+ = ((A^T A) * (B^T B))^+ (A \odot B)^T. \quad (2.5)$$

Hint: You can use the equation

$$(A \odot B)^T (A \odot B) = A^T A * B^T B. \quad (2.6)$$

Solution. To prove the claim, we need to show that setting $M = A \odot B$ and $M^+ = ((A^T A) * (B^T B))^+ (A \odot B)^T$ satisfies (2.1)–(2.2). We start from (2.1):

$$\begin{aligned} (A \odot B)(A \odot B)^+(A \odot B) &= (A \odot B)((A^T A) * (B^T B))^+ (A \odot B)^T (A \odot B) \\ &= (A \odot B)((A^T A) * (B^T B))^+ ((A^T A) * (B^T B)) \\ &= (A \odot B), \end{aligned}$$

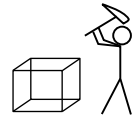
where the least equality is due to the fact that $(A^T A) * (B^T B)$ is invertible and hence $((A^T A) * (B^T B))^+ = ((A^T A) * (B^T B))^{-1}$.

Equation (2.2) follows similarly. For (2.3), we write

$$\begin{aligned} ((A \odot B)(A \odot B)^+)^T &= ((A \odot B)((A^T A) * (B^T B))^+ (A \odot B)^T)^T \\ &= (((A^T A) * (B^T B))^+ (A \odot B)^T)^T (A \odot B)^T \\ &= (A \odot B) \left(((A^T A) * (B^T B))^+ \right)^T (A \odot B)^T \\ &= (A \odot B) ((A^T A) * (B^T B))^+ (A \odot B)^T \\ &= (A \odot B)(A \odot B)^+, \end{aligned}$$

where the penultimate equation follows from the fact that $(A^T A) * (B^T B)$ is a symmetric matrix. Equation (2.4) follows more easily from the properties of the pseudo-inverse:

$$\begin{aligned} ((A \odot B)^+(A \odot B))^T &= (((A^T A) * (B^T B))^+ (A \odot B)^T (A \odot B))^T \\ &= (((A \odot B)^T (A \odot B))^+ (A \odot B)^T (A \odot B))^T \\ &= ((A \odot B)^T (A \odot B))^+ (A \odot B)^T (A \odot B) \\ &= (A \odot B)^+(A \odot B). \end{aligned}$$



Problem 3 (CP decomposition). Let

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}.$$

- Calculate $\mathbf{a}_1 \circ \mathbf{b}_1 \circ \mathbf{c}_1$.
- Calculate $\mathbf{a}_2 \circ \mathbf{b}_2 \circ \mathbf{c}_2$.
- Calculate $\llbracket \mathbf{A}, \mathbf{B}, \mathbf{C} \rrbracket$.

Solution.

- Calling the result $\mathcal{T}^{(1)}$, the frontal slices are

$$\mathbf{T}_1^{(1)} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \quad \text{and} \quad \mathbf{T}_2^{(1)} = \begin{pmatrix} -2 & -4 & -6 \\ -4 & -8 & -12 \\ -6 & -12 & -18 \end{pmatrix}.$$

- Calling the result $\mathcal{T}^{(2)}$, the frontal slices are

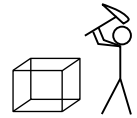
$$\mathbf{T}_1^{(2)} = \begin{pmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{pmatrix} \quad \text{and} \quad \mathbf{T}_2^{(2)} = \begin{pmatrix} -8 & -16 & -24 \\ -10 & -20 & -30 \\ -12 & -24 & -36 \end{pmatrix}.$$

- The result is

$$\llbracket \mathbf{A}, \mathbf{B}, \mathbf{C} \rrbracket = \mathbf{a}_1 \circ \mathbf{b}_1 \circ \mathbf{c}_1 + \mathbf{a}_2 \circ \mathbf{b}_2 \circ \mathbf{c}_2$$

with

$$\llbracket \mathbf{A}, \mathbf{B}, \mathbf{C} \rrbracket_1 = \begin{pmatrix} 5 & 10 & 15 \\ 7 & 14 & 21 \\ 9 & 18 & 27 \end{pmatrix}$$
$$\llbracket \mathbf{A}, \mathbf{B}, \mathbf{C} \rrbracket_2 = \begin{pmatrix} -10 & -20 & -30 \\ -14 & -28 & -42 \\ -18 & -36 & -54 \end{pmatrix}.$$



Problem 4 (Uniqueness of a rank decomposition). In the lectures we saw a tensor $\mathcal{T} \in \mathbb{R}^{2 \times 2 \times 2}$,

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

that has real tensor rank 3. One factorization that obtains this rank 3 is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

What can you say about the uniqueness of this factorization?

Solution. First, $\text{rank}(\mathbf{A} \odot \mathbf{B}) = \text{rank}(\mathbf{A} \odot \mathbf{C}) = \text{rank}(\mathbf{B} \odot \mathbf{C}) = 3$, so the first necessary condition is fulfilled.

On the other hand, it is easy to see that the k -rank of the factor matrices is 2 for all of them. Hence

$$k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{C}} = 6 < 2 \cdot 3 + 2 = 8,$$

and the sufficient condition of the uniqueness isn't fulfilled. Reading Kolda & Bader, we learn that this sufficient condition is also necessary for tensors with rank 2 or 3. Hence the factorization is not unique.