D5: Databases and Information Systems
Tensors in Data Analysis, WS 2017–18
Homework #2: CP decomposition and tensor rank
Tutorial: **11 October 2017** at 14:30

Problem 1 (Khatri–Rao is associative). Let $\boldsymbol{A} \in \mathbb{R}^{I\times L}$, $\boldsymbol{B} \in \mathbb{R}^{J\times L}$, and $\boldsymbol{C} \in \mathbb{R}^{K\times L}$. Show that

$$(\boldsymbol{A} \odot \boldsymbol{B}) \odot \boldsymbol{C} = \boldsymbol{A} \odot (\boldsymbol{B} \odot \boldsymbol{C})\ .$$

Solution. Let $\boldsymbol{D} = (\boldsymbol{A} \odot \boldsymbol{B})$ so that $d_{(i\cdot j)\ell} = a_{i\ell}b_{j\ell}$. Then

$$\boldsymbol{D} \odot \boldsymbol{C} = [\boldsymbol{d}_1 \otimes \boldsymbol{c}_1\ \boldsymbol{d}_2 \otimes \boldsymbol{c}_2\ \dots\ \boldsymbol{d}_L \otimes \boldsymbol{c}_K]$$

and

$$(\boldsymbol{D} \odot \boldsymbol{C})_{(i\cdot j\cdot k)\ell} = d_{(i\cdot j)\ell}c_{k\ell} = a_{i\ell}b_{j\ell}c_{k\ell} = a_{i\ell}(\boldsymbol{b}_\ell \otimes \boldsymbol{c}_\ell)_{(j\cdot k)\ell} = \big(\boldsymbol{A} \odot (\boldsymbol{B} \odot \boldsymbol{C})\big)_{(i\cdot j\cdot k)\ell}\ ,$$

proving the claim.

Problem 2 (Khatri–Rao and pseudo-inverse). Recall that the Moore–Penrose pseudo-inverse of a matrix $\boldsymbol{M}$ is a matrix $\boldsymbol{M}^+$ such that

$$\boldsymbol{M}\boldsymbol{M}^+\boldsymbol{M} = \boldsymbol{M} \tag{2.1}$$

$$\boldsymbol{M}^+\boldsymbol{M}\boldsymbol{M}^+ = \boldsymbol{M}^+ \tag{2.2}$$

$$(\boldsymbol{M}\boldsymbol{M}^+)^T = \boldsymbol{M}\boldsymbol{M}^+ \tag{2.3}$$

$$(\boldsymbol{M}^+\boldsymbol{M})^T = \boldsymbol{M}^+\boldsymbol{M}\ . \tag{2.4}$$

Let $\boldsymbol{A} \in \mathbb{R}^{I \times K}$ and $\boldsymbol{B} \in \mathbb{R}^{J \times K}$ be such that $K < \min\{I, J\}$ and $\operatorname{rank}\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big) = K$. Show that

$$(\boldsymbol{A} \odot \boldsymbol{B})^+ = \big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+(\boldsymbol{A} \odot \boldsymbol{B})^T\ . \tag{2.5}$$

Hint: You can use the equation

$$(\boldsymbol{A} \odot \boldsymbol{B})^T(\boldsymbol{A} \odot \boldsymbol{B}) = \boldsymbol{A}^T\boldsymbol{A} * \boldsymbol{B}^T\boldsymbol{B}\ . \tag{2.6}$$

Solution. To prove the claim, we need to show that setting $\boldsymbol{M} = \boldsymbol{A} \odot \boldsymbol{B}$ and $\boldsymbol{M}^+ = \big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+(\boldsymbol{A} \odot \boldsymbol{B})^T$ satisfies (2.1)–(2.2). We start from (2.1):

$$\begin{aligned}
(\boldsymbol{A} \odot \boldsymbol{B})(\boldsymbol{A} \odot \boldsymbol{B})^+(\boldsymbol{A} \odot \boldsymbol{B}) &= (\boldsymbol{A} \odot \boldsymbol{B})\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+(\boldsymbol{A} \odot \boldsymbol{B})^T(\boldsymbol{A} \odot \boldsymbol{B}) \\
&= (\boldsymbol{A} \odot \boldsymbol{B})\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big) \\
&= (\boldsymbol{A} \odot \boldsymbol{B})\ ,
\end{aligned}$$

where the least equality is due to the fact that $(\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})$ is invertible and hence $\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+ = \big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^{-1}$.

Equation (2.2) follows similarly. For (2.3), we write

$$\begin{aligned}
\big((\boldsymbol{A} \odot \boldsymbol{B})(\boldsymbol{A} \odot \boldsymbol{B})^+\big)^T &= \Big((\boldsymbol{A} \odot \boldsymbol{B})\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+(\boldsymbol{A} \odot \boldsymbol{B})^T\Big)^T \\
&= \Big(\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+(\boldsymbol{A} \odot \boldsymbol{B})^T\Big)^T(\boldsymbol{A} \odot \boldsymbol{B})^T \\
&= (\boldsymbol{A} \odot \boldsymbol{B})\Big(\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+\Big)^T(\boldsymbol{A} \odot \boldsymbol{B})^T \\
&= (\boldsymbol{A} \odot \boldsymbol{B})\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+(\boldsymbol{A} \odot \boldsymbol{B})^T \\
&= (\boldsymbol{A} \odot \boldsymbol{B})(\boldsymbol{A} \odot \boldsymbol{B})^+\ ,
\end{aligned}$$

where the penultimate equation follows from the fact that $(\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})$ is a symmetric matrix. Equation (2.4) follows more easily from the properties of the pseudo-inverse:

$$\begin{aligned}
\big((\boldsymbol{A} \odot \boldsymbol{B})^+(\boldsymbol{A} \odot \boldsymbol{B})\big)^T &= \Big(\big((\boldsymbol{A}^T\boldsymbol{A}) * (\boldsymbol{B}^T\boldsymbol{B})\big)^+(\boldsymbol{A} \odot \boldsymbol{B})^T(\boldsymbol{A} \odot \boldsymbol{B})\Big)^T \\
&= \Big(\big((\boldsymbol{A} \odot \boldsymbol{B})^T(\boldsymbol{A} \odot \boldsymbol{B})\big)^+(\boldsymbol{A} \odot \boldsymbol{B})^T(\boldsymbol{A} \odot \boldsymbol{B})\Big)^T \\
&= \big((\boldsymbol{A} \odot \boldsymbol{B})^T(\boldsymbol{A} \odot \boldsymbol{B})\big)^+(\boldsymbol{A} \odot \boldsymbol{B})^T(\boldsymbol{A} \odot \boldsymbol{B}) \\
&= (\boldsymbol{A} \odot \boldsymbol{B})^+(\boldsymbol{A} \odot \boldsymbol{B})\ .
\end{aligned}$$

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Problem 3 (CP decomposition). Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{C} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}.$$

a) Calculate $\boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1$.

b) Calculate $\boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2$.

c) Calculate $[\![\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]\!]$.

Solution.

a) Calling the result $\boldsymbol{\mathcal{T}}^{(1)}$, the frontal slices are

$$\boldsymbol{T}_1^{(1)} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \quad \text{and} \quad \boldsymbol{T}_2^{(1)} = \begin{pmatrix} -2 & -4 & -6 \\ -4 & -8 & -12 \\ -6 & -12 & -18 \end{pmatrix}.$$

b) Calling the result $\boldsymbol{\mathcal{T}}^{(2)}$, the frontal slices are

$$\boldsymbol{T}_1^{(2)} = \begin{pmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{pmatrix} \quad \text{and} \quad \boldsymbol{T}_2^{(2)} = \begin{pmatrix} -8 & -16 & -24 \\ -10 & -20 & -30 \\ -12 & -24 & -36 \end{pmatrix}.$$

c) The result is

$$[\![\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]\!] = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2$$

with

$$[\![\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]\!]_1 = \begin{pmatrix} 5 & 10 & 15 \\ 7 & 14 & 21 \\ 9 & 18 & 27 \end{pmatrix}$$

$$[\![\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]\!]_2 = \begin{pmatrix} -10 & -20 & -30 \\ -14 & -28 & -42 \\ -18 & -36 & -54 \end{pmatrix}.$$

Problem 4 (Uniqueness of a rank decomposition). In the lectures we saw a tensor $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{2\times 2\times 2}$,

$$\boldsymbol{T}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{T}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

that has real tensor rank 3. One factorization that obtains this rank 3 is

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{C} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} .$$

What can you say about the uniqueness of this factorization?

Solution. First, $\operatorname{rank}(\boldsymbol{A} \odot \boldsymbol{B}) = \operatorname{rank}(\boldsymbol{A} \odot \boldsymbol{C}) = \operatorname{rank}(\boldsymbol{B} \odot \boldsymbol{C}) = 3$, so the first necessary condition is fulfilled.

On the other hand, it is easy to see that the k-rank of the factor matrices is 2 for all of them. Hence

$$k_{\boldsymbol{A}} + k_{\boldsymbol{A}} + k_{\boldsymbol{A}} = 6 < 2 \cdot 3 + 2 = 8 ,$$

and the sufficient condition of the uniqueness isn't fulfilled. Reading Kolda & Bader, we learn that this sufficient condition is also necessary for tensors with rank 2 or 3. Hence the factorization is not unique.