# SCL(T): Simple Clause Learning modulo Theories

## Input:

- Set of clauses: $N = \{C_1\colon x \geq y \parallel \neg P(x,y) \vee Q(x),\ \ C_2\colon u \leq v \parallel \neg P(u,v) \vee \neg Q(u)\}$

## State:

- Set of constants: $B = \{a, b\}$
- Ground partial model assumption (Trail): $M = P(a,b)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}, a \geq b,$
- Conflict: $(M;\ C_i \cdot \sigma)$
- Learned Clauses (not ground!): $U = \{\ C_4\colon x \geq y \wedge x \leq y \parallel \neg P(x,y)\ \}$

## Techniques:

- Model assumption build via Decisions & Propagations
- Conflicts resolved via Hierarchic Resolution
- Conflicting models blocked via learned clauses

## Results:

- Derive empty clause
- $M$ corresponds to valid interpretation
- Need more constants

# SCL(T) Example

$B = \{a\}$

$N = \{ C_1 \colon x \geq y \parallel \neg P(x,y) \vee Q(x), \quad C_2 \colon u \leq v \parallel \neg P(u,v) \vee \neg Q(u) \}$

$U = \{ \}$

$\Rightarrow^{Decide}_{SCL(T)} \quad P(a,a)^1,$

$P(x,y)\{x \mapsto a, y \mapsto a\} \Rightarrow P(a,a)$

**Decide:**

$M \Rightarrow_{SCL(T)} M, L\,\sigma^{k+1}, \Lambda\sigma$

Provided:

- $|L| \in atoms(N \cup U)$,
- $|K| \in atoms(N \cup U)$ for all $|K| \in \Lambda$,
- $\sigma$ is grounding over $B$,
- $L\sigma$ is undefined in $M$,
- $k$ decisions in $M$,
- $\Lambda\sigma$ is satisfiable (T) under $M$, i.e.,

$$\vDash_T adiff(B) \wedge M \wedge \Lambda\sigma$$

$B = \{a\}$

# SCL(T) Example

$N = \{C_1 : x \geq y \parallel \neg P(x,y) \vee Q(x),\ \ C_2 : u \leq v \parallel \neg P(u,v) \vee \neg Q(u)\ \}$

$U = \{\ \}$

$\Rightarrow_{SCL(T)}^{Decide}\quad P(a,a)^1,$

$\Rightarrow_{SCL(T)}^{Propagate}\quad P(a,a)^1,\ Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}},\ a \geq a,$

$C_1 \cdot \{x \mapsto a, y \mapsto a\}$:

$a \geq a \parallel \neg P(a,a) \vee Q(a)$

$\top \quad \bot \quad ?$

**Propagate:**

$M \ \Rightarrow_{SCL(T)} M, L\,\sigma^{\Lambda \parallel C}, \Lambda\sigma$

Provided:

- Theory constraint is satisfiable ($\top$) under $M$, i.e.,

  $\vDash_T adiff(B) \wedge M \wedge \Lambda\sigma$
- One foreground literal $L\sigma$ is undefined ($?$) in $M$
- All other foreground literals are false ($\bot$) under $M$

$B = \{a\}$

# SCL(T) Example

$N = \{ C_1\colon x \geq y \parallel \neg P(x,y) \vee Q(x),\ C_2\colon u \leq v \parallel \neg P(u,v) \vee \neg Q(u) \}$

$U = \{ \}$

$\Rightarrow_{SCL(T)}^{Decide} \quad P(a,a)^1,$

$\Rightarrow_{SCL(T)}^{Propagate} \quad P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}, a \geq a,$

$\Rightarrow_{SCL(T)}^{Conflict} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}, a \geq a; C_2 \cdot \sigma)$

$C_2 \cdot \{u \mapsto a, v \mapsto a\}$:

$a \leq a \parallel \neg P(a,a) \vee \neg Q(a)$

$\top \quad \bot \quad \bot$

**Conflict:**

$M \Rightarrow_{SCL(T)} (M; \Lambda \parallel C \cdot \sigma)$

Provided:

- Theory constraint is satisfiable ($\top$) under $M$, i.e.,
  $\vDash_T adiff(B) \wedge M \wedge \Lambda\sigma$
- All foreground literals are false ($\bot$) under $M$

# SCL(T) Example

$B = \{a\}$

$N = \{C_1 : x \geq y \parallel \neg P(x,y) \vee Q(x),\ \ C_2 : u \leq v \parallel \neg P(u,v) \vee \neg Q(u)\ \}$

$U = \{\ \}$

$\Rightarrow^{Decide}_{SCL(T)}\quad P(a,a)^1,$

$\Rightarrow^{Propagate}_{SCL(T)}\quad P(a,a)^1,\ Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}},\ \ a \geq a,$

$\Rightarrow^{Conflict}_{SCL(T)}\quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}, a \geq a; C_2 \cdot \sigma)$

$\Rightarrow^{Skip}_{SCL(T)}\quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_2 \cdot \sigma)$

$\overbrace{u \leq v \parallel \neg P(u,v) \vee \neg Q(u)}^{C_2} \cdot \overbrace{\{u \mapsto a, v \mapsto a\}}^{\sigma}$

**Skip:** $(M, L; \Lambda \parallel C \cdot \sigma)$

$\Rightarrow_{SCL(T)} (M; \Lambda \parallel C \cdot \sigma)$

Provided:

- $|L|$ does not appear in $C \cdot \sigma$

# SCL(T) Example

$B = \{a\}$

$N = \{ C_1 \colon x \geq y \parallel \neg P(x,y) \lor Q(x), \quad C_2 \colon u \leq v \parallel \neg P(u,v) \lor \neg Q(u) \ \}$

$U = \{\ \}$

$\eta = mgu\big(Q(x), Q(u)\big) = \{u \mapsto x\}$

$\underbrace{x \leq v \land x \geq y \parallel \neg P(x,y) \lor \neg P(x,v)}_{C_3 = (\Lambda \land \Lambda' \parallel D \lor D')\eta} \cdot \underbrace{\{x \mapsto a, y \mapsto a, v \mapsto a\}}_{\sigma' = \sigma\rho}$

$\Rightarrow_{SCL(T)}^{Skip} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_2 \cdot \sigma)$

$\Rightarrow_{SCL(T)}^{Resolve} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_3 \cdot \sigma')$

$\overbrace{u \leq v \parallel \neg P(u,v) \lor \neg Q(u)}^{C_2} \cdot \overbrace{\{u \mapsto a, v \mapsto a\}}^{\sigma}$

$\oplus$

$\underbrace{x \geq y \parallel \neg P(x,y) \lor \quad Q(x)}_{C_1} \cdot \underbrace{\{x \mapsto a, y \mapsto a\}}_{\rho}$

**Resolve:** $\big(M, L'\rho^{C' \cdot \rho}; \Lambda \parallel C \cdot \sigma\big)$

$\Rightarrow_{SCL(T)} \big(M, L'\rho^{C' \cdot \rho}; \Lambda^* \parallel D^* \cdot \sigma\rho\big)$

Provided:

- $C'\rho = \Lambda'\rho \parallel D'\rho \lor L'\rho$
- $C\sigma = D\sigma \lor L\sigma$
- $L'\rho = comp(L\sigma)$
- $\eta = mgu(L', comp(L))$
- $\Lambda^* \parallel D^* = (\Lambda \land \Lambda' \parallel D \lor D')\ \eta$

# SCL(T) Example

$B = \{a\}$

$N = \{ C_1\colon x \geq y \parallel \neg P(x,y) \vee Q(x),\quad C_2\colon u \leq v \parallel \neg P(u,v) \vee \neg Q(u) \ \}$

$U = \{\,\}$

$\eta = mgu\big(Q(x), Q(u)\big) = \{u \mapsto x\}$

$\underbrace{x \leq v \wedge x \geq y \parallel \neg P(x,y) \vee \neg P(x,v)}_{C_3 = (\Lambda \wedge \Lambda' \parallel D \vee D')\eta} \cdot \underbrace{\{x \mapsto a, y \mapsto a, v \mapsto a\}}_{\sigma' = \sigma\rho}$

$\Rightarrow^{Skip}_{SCL(T)} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_2 \cdot \sigma)$

$\Rightarrow^{Resolve}_{SCL(T)} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_3 \cdot \sigma')$

$\overbrace{x \leq v \wedge x \geq y \parallel \neg P(x,y) \vee \neg P(x,v)}^{C_3} \cdot \underbrace{\{x \mapsto a, y \mapsto a, v \mapsto a\}}_{\sigma'}$

$\Rightarrow \ a \leq a \wedge a \geq a \parallel \neg P(a,a) \vee \neg P(a,a)$

**Resolve:** $(M, L'\rho^{C' \cdot \rho}; \Lambda \parallel C \cdot \sigma)$

$\Rightarrow_{SCL(T)} (M, L'\rho^{C' \cdot \rho}; \Lambda^* \parallel D^* \cdot \sigma\rho)$

Provided:

- $C'\rho = \Lambda'\rho \parallel D'\rho \vee L'\rho$
- $C\sigma = D\sigma \vee L\sigma$
- $L'\rho = comp(L\sigma)$
- $\eta = mgu(L', comp(L))$
- $\Lambda^* \parallel D^* = (\Lambda \wedge \Lambda' \parallel D \vee D')\,\eta$

# SCL(T) Example

$B = \{a\}$

$N = \{\, C_1\colon\ x \geq y \parallel \neg P(x,y) \lor Q(x),\quad C_2\colon u \leq v \parallel \neg P(u,v) \lor \neg Q(u) \,\}$

$U = \{\,\}$

$\eta = mgu\big(\neg P(x,y), \neg P(x,v)\big) = \{v \mapsto y\}$

$$\underbrace{x \leq y \land x \geq y \parallel \neg P(x,y)}_{C_4} \cdot \underbrace{\{x \mapsto a, y \mapsto a, v \mapsto a\}}_{\sigma'}$$

$\Rightarrow^{Skip}_{SCL(T)}\ (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_2 \cdot \sigma)$

$\Rightarrow^{Resolve}_{SCL(T)}\ (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_3 \cdot \sigma')$

$\Rightarrow^{Factorize}_{SCL(T)}\ (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_4 \cdot \sigma')$

$$\overbrace{x \leq v \land x \geq y \parallel \neg P(x,y) \lor \neg P(x,v)}^{C_3}$$
$$\cdot \underbrace{\{x \mapsto a, y \mapsto a, v \mapsto a\}}_{\sigma'}$$

$\Rightarrow\ a \leq a \land a \geq a \parallel \neg P(a,a) \lor \neg P(a,a)$

**Factorize:** $(M; \Lambda \parallel C \lor L \lor L' \cdot \sigma)$

$\Rightarrow_{SCL(T)} (M; (\Lambda \parallel C \lor L)\eta \cdot \sigma)$

Provided:

- $L\sigma = L'\sigma$
- $\eta = mgu(L, L')$

# SCL(T) Example

$B = \{a\}$

$N = \{ C_1 : x \geq y \parallel \neg P(x,y) \vee Q(x),\ \ C_2 : u \leq v \parallel \neg P(u,v) \vee \neg Q(u) \ \}$

$U = \{ \}$

$\Rightarrow^{Decide}_{SCL(T)} \quad P(a,a)^1,$

$\Rightarrow^{Propagate}_{SCL(T)} \quad P(a,a)^1,\ Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}},\ \ a \geq a,$

$\Rightarrow^{Conflict}_{SCL(T)} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}, a \geq a; C_2 \cdot \sigma)$

$\Rightarrow^{Skip}_{SCL(T)} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_2 \cdot \sigma)$

$\Rightarrow^{Resolve}_{SCL(T)} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_3 \cdot \sigma')$

$\Rightarrow^{Factorize}_{SCL(T)} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_4 \cdot \sigma')$

$$\underbrace{x \leq y \wedge x \geq y \parallel \neg P(x,y)}_{C_4} \cdot \underbrace{\{x \mapsto a, y \mapsto a\}}_{\sigma'}$$

$\Rightarrow\ a \leq a \wedge a \geq a \parallel \neg P(a,a)$

**Factorize:** $(M; \Lambda \parallel C \vee L \vee L' \cdot \sigma)$

$\quad \Rightarrow_{SCL(T)} (M; (\Lambda \parallel C \vee L)\eta \cdot \sigma)$

Provided:

- $L\sigma = L'\sigma$
- $\eta = mgu(L, L')$

# SCL(T) Example

$B = \{a\}$

$N = \{ C_1 : x \geq y \parallel \neg P(x,y) \vee Q(x), \quad C_2 : u \leq v \parallel \neg P(u,v) \vee \neg Q(u) \ \}$

$U = \{ \}$

$\Rightarrow_{SCL(T)}^{Decide} \quad P(a,a)^1,$

$\Rightarrow_{SCL(T)}^{Propagate} \quad P(a,a)^1,\ Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}, \quad a \geq a,$

$\Rightarrow_{SCL(T)}^{Conflict} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}, a \geq a; C_2 \cdot \sigma)$

$\Rightarrow_{SCL(T)}^{Skip} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_2 \cdot \sigma)$

$\Rightarrow_{SCL(T)}^{Resolve} \quad (P(a,a)^1, Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}}; C_3 \cdot \sigma')$

$\Rightarrow_{SCL(T)}^{Factorize} \quad (P(a,a)^1, \boxed{Q(a)^{C_1 \cdot \{x \mapsto a, y \mapsto a\}};} C_4 \cdot \sigma')$

$\Rightarrow_{SCL(T)}^{Skip} \quad (P(a,a)^1; C_4 \cdot \sigma')$

$\underbrace{x \leq y \wedge x \geq y \parallel \neg P(x,y)}_{C_4} \cdot \underbrace{\{x \mapsto a, y \mapsto a\}}_{\sigma'}$

$\Rightarrow \ a \leq a \wedge a \geq a \parallel \neg P(a,a)$

**Skip:** $(M, L; \Lambda \parallel C \cdot \sigma)$

$\Rightarrow_{SCL(T)} (M; \Lambda \parallel C \cdot \sigma)$

Provided:

- $|L|$ does not appear in $C \cdot \sigma$

# SCL(T) Example

$B = \{a\}$

$N = \{C_1: x \geq y \parallel \neg P(x,y) \vee Q(x), \quad C_2: u \leq v \parallel \neg P(u,v) \vee \neg Q(u) \quad \}$

$U = \{\,\}\ C_4: x \geq y \wedge x \leq y \parallel \neg P(x,y) \quad \}$

$\Rightarrow_{SCL(T)}^{Decide} \quad P(a,a)^1,$

$\Rightarrow_{SCL(T)}^{Propagate} \quad P(a,a)^1,\ Q(a)^{C_1\cdot\{x\mapsto a, y\mapsto a\}}, \quad a \geq a,$

$\Rightarrow_{SCL(T)}^{Conflict} \quad (P(a,a)^1, Q(a)^{C_1\cdot\{x\mapsto a, y\mapsto a\}}, a \geq a; C_2 \cdot \sigma)$

$\Rightarrow_{SCL(T)}^{Skip} \quad (P(a,a)^1, Q(a)^{C_1\cdot\{x\mapsto a, y\mapsto a\}}; C_2 \cdot \sigma)$

$\Rightarrow_{SCL(T)}^{Resolve} \quad (P(a,a)^1, Q(a)^{C_1\cdot\{x\mapsto a, y\mapsto a\}}; C_3 \cdot \sigma')$

$\Rightarrow_{SCL(T)}^{Factorize} \quad (P(a,a)^1, Q(a)^{C_1\cdot\{x\mapsto a, y\mapsto a\}}; C_4 \cdot \sigma')$

$\Rightarrow_{SCL(T)}^{Skip} \quad (P(a,a)^1; C_4 \cdot \sigma')$

$\Rightarrow_{SCL(T)}^{Backtrack} \quad \neg P(a,a)^{C_4\cdot\{x\mapsto a, y\mapsto a\}}, \ a \geq a, \ a \leq a$

$\overbrace{x \leq y \wedge x \geq y \parallel \neg P(x,y)}^{C_4} \cdot \overbrace{\{x \mapsto a, y \mapsto a\}}^{\sigma'}$

**Backtrack:** $(M, K^{i+1}, M'; C \cdot \sigma)$

$\Rightarrow_{SCL(T)} M, L^{C\cdot\sigma}, \Lambda$

Provided:

- $C\sigma = \Lambda \parallel D \vee L$
- $L$ is of level $k > i$, and $D$ is of level $i$
- $U := U \cup \{C\}$

# SCL(T) Example

$B = \{a\}$

$N = \{ C_1\colon x \geq y \parallel \neg P(x,y) \lor Q(x),\quad C_2\colon u \leq v \parallel \neg P(u,v) \lor \neg Q(u) \ \}$

$U = \{ \quad C_4\colon x \geq y \land x \leq y \parallel \neg P(x,y) \quad \}$

$\Rightarrow^{Decide}_{SCL(T)} \quad P(a,a)^1,$

$\Rightarrow^{Propagate}_{SCL(T)} \quad P(a,a)^1,\ Q(a)^{C_1\cdot\{x\mapsto a, y\mapsto a\}},\quad a \geq a,$

$\Rightarrow^{Conflict}_{SCL(T)} \quad (P(a,a)^1, Q(a)^{C_1\cdot\{x\mapsto a, y\mapsto a\}}, a \geq a; C_2 \cdot \sigma)$

$\Rightarrow^{Skip}_{SCL(T)} \quad (P(a,a)^1, Q(a)^{C_1\cdot\{x\mapsto a, y\mapsto a\}}; C_2 \cdot \sigma)$

$\Rightarrow^{Resolve}_{SCL(T)} \quad (P(a,a)^1, Q(a)^{C_1\cdot\{x\mapsto a, y\mapsto a\}}; C_3 \cdot \sigma')$

$\Rightarrow^{Factorize}_{SCL(T)} \quad (P(a,a)^1, Q(a)^{C_1\cdot\{x\mapsto a, y\mapsto a\}}; C_4 \cdot \sigma')$

$\Rightarrow^{Skip}_{SCL(T)} \quad (P(a,a)^1; C_4 \cdot \sigma')$

$\Rightarrow^{Backtrack}_{SCL(T)} \quad \neg P(a,a)^{C_4\cdot\{x\mapsto a, y\mapsto a\}},\ a \geq a,\ a \leq a$

$\Rightarrow^{Decide}_{SCL(T)} \quad \dots, Q(a)^1$

# SCL(T) Example 2

$B = \{a, b, c\}$

$N = \{C_1: x \geq y \parallel \neg P(x,y) \vee Q(x),\ \ C_2: u \leq v \parallel \neg P(u,v) \vee \neg Q(u),\ \ C_3: x \neq y \parallel P(x,y)\}$

$U = \{\}$

$\Rightarrow^{Propagate}_{SCL(T)}\ P(a,b)^{C_3 \cdot \{x \mapsto a, y \mapsto b\}}, a \neq b,$

$\Rightarrow^{Propagate}_{SCL(T)}\ \ldots, P(b,c)^{C_3 \cdot \{x \mapsto b, y \mapsto c\}}, b \neq c,$

$\Rightarrow^{Propagate}_{SCL(T)}\ \ldots, Q(b)^{C_1 \cdot \{x \mapsto b, y \mapsto a\}}, b \geq a$

$\Rightarrow^{Conflict}_{SCL(T)}\ (\ldots, Q(b)^{C_1 \cdot \{x \mapsto b, y \mapsto a\}}; C_2 \cdot \sigma)$

$\Rightarrow^{*\ Resolve}_{SCL(T)}\ (\emptyset; C_4 \cdot \sigma)$

$$\overbrace{x \neq y, x \neq v, x \geq y, x \leq v \parallel \bot}^{C_4} \cdot \underbrace{\{x \mapsto b, y \mapsto a, v \mapsto c\}}_{\sigma}$$

$$b \neq a, b \neq c, b \geq a, b \leq c \parallel \bot$$

$\top$ $\bot$