Exercise 10.1.

Choose $\beta = P(f(f(a)))$ and $\{P(a), P(b), Q(a), Q(b), P(f(a)), P(f(b)), Q(f(a)), Q(f(b))\}$

set of ground literals $\prec_\beta \beta$

$(\varepsilon; N; \emptyset; 0; \top)$

$$\Rightarrow_{SCL}^{\text{Propagate}^*} (\overbrace{Q(a)^{Q(a)}, \neg Q(f(a))^{\neg Q(f(x)) \lor \neg Q(x) \cdot \{x \mapsto a\}}, \neg P(a)^{Q(f(x)) \lor \neg P(x) \cdot \{x \mapsto a\}}, P(b)^{P(a) \lor P(b)}, P(f(b))^{\neg P(x) \lor P(f(x)) \cdot \{x \mapsto b\}}}^{\Gamma_1}, N, \emptyset, 0, \top)$$

$$\Rightarrow_{SCL}^{\text{Conflict}} (\Gamma_1, N, \emptyset, 0, \neg P(x) \lor \neg P(f(x)) \lor Q(f(a)) \cdot \{x \mapsto b\})$$

$$\Rightarrow_{SCL}^{\text{Resolve, Skip}} (\ldots, P(b)^{P(a) \lor P(b)}, N, \emptyset, 0, \neg P(x) \lor \neg P(x) \lor Q(f(a)) \cdot \{x \mapsto b\})$$

$$\Rightarrow_{SCL}^{\text{Factorize}} (\ldots, P(b)^{P(a) \lor P(b)}, N, \emptyset, 0, \neg P(x) \lor Q(f(a)) \cdot \{x \mapsto b\})$$

$$\Rightarrow_{SCL}^{\text{Resolve, Skip}} (\ldots, \neg P(a)^{Q(f(x)) \lor \neg P(x) \cdot \{x \mapsto a\}}, N, \emptyset, 0, P(a) \lor Q(f(a)))$$

$$\Rightarrow_{SCL}^{\text{Resolve, Skip}} (\ldots, \neg Q(f(a))^{\neg Q(f(x)) \lor \neg Q(x) \cdot \{x \mapsto a\}}, N, \emptyset, 0, Q(f(a)) \lor Q(f(a)))$$

$$\Rightarrow_{SCL}^{\text{Factorize}} (\text{——— '' ———}, Q(f(a)))$$

$$\Rightarrow_{SCL}^{\text{Resolve, Skip}} (Q(a)^{Q(a)}, N, \emptyset, 0, \neg Q(a))$$

$$\Rightarrow_{SCL}^{\text{Resolve, Skip}} (\varepsilon, N, \emptyset, 0, \bot)$$

Exercise 10.2;

$$E \begin{cases} (1) & f(a, g(a)) \approx f(b, g(b)), \\ (2) & g(a) \approx h(c), \\ (3) & h(d) \approx g(b), \\ (4) & d \approx c, \\ (5) & f(a, h(d)) \approx f(h(d), a), \\ (6) & f(b, g(b)) \not\approx f(h(c), a) \end{cases}$$

(Only highlight the equations we changed/added by number)

~~First~~

First Flattening of E

new

E

$\Rightarrow_{CCF} E \wedge (1) f(a, e_1) \approx f(b, g(b)) \wedge (2) e_1 \approx h(c) \wedge \underline{(7)\ g(a) \approx e_1}$

$\Rightarrow_{CCF} E_1 \wedge (1) f(a, e_1) \approx f(b, e_2) \wedge (3) h(d) \approx e_2 \wedge \underline{(8)\ g(b) \approx e_2}$

$\Rightarrow_{CCF} E_2 \wedge (1)\ e_3 \approx f(b, e_2) \wedge \underline{(9)\ f(a, e_1) \approx e_3}$

$\Rightarrow_{CCF} E_3 \wedge (3)\ e_4 \approx e_2 \wedge (5) f(a, e_4) \approx f(e_4, a) \wedge \underline{(10)\ h(d) \approx e_4}.$

$\Rightarrow_{CCF} E_4 \wedge (5)\ e_5 \approx f(e_4, a) \wedge \underline{(11)\ f(a, e_4) \approx e_5}$

That means we get

$E' = \{(3)\ e_4 \approx e_2,\ (4)\ d \approx c\}$

$R_0 = \{(1)\ f(b, e_2) \to e_3,\ (2)\ h(c) \to e_1,\ (3)\ f(e_4, a) \to e_5,$
$(4)\ g(a) \to e_1,\ (5)\ g(b) \to e_2,\ (6)\ f(a, e_1) \to e_3,$
$(7)\ h(d) \to e_4,\ (8)\ f(a, e_4) \to e_5\}$

As our ordering, we choose:

$a > b > c > d > e_1 > e_2 > e_3 > e_4 > e_5$

⚠ R_i is the set of rules after $i \Rightarrow_{CC}$ steps

(E', R)

$\overset{\text{Orient }(E'.3)}{\underset{CC}{\Longrightarrow}} \; (E' \setminus \{(3)\, e_4 \approx e_2\}, R_0 \cup \{(9)\, e_2 \to e_4\})$

$\overset{\text{Orient }(E'.4)}{\underset{CC}{\Longrightarrow}} \; (\emptyset, R_1 \cup \{(10)\, c \to d\})$

$\overset{\text{Collapse }(R.1, R.9)}{\underset{CC}{\Longrightarrow}} \; (\emptyset, (R_2 \setminus \{(1)\, f(b, e_2) \to e_3\}) \cup \{(11)\, f(b, e_4) \to e_3\})$

$\overset{\text{Collapse }(R.2, R.10)}{\underset{CC}{\Longrightarrow}} \; (\emptyset, (R_3 \setminus \{(2)\, h(c) \to e_1\}) \cup \{(12)\, h(d) \to e_1\})$

$\overset{\text{Deduce }(R.7, R.12)}{\underset{CC}{\Longrightarrow}} \; (\{(12)\, e_1 \approx e_4\}, R_4 \setminus \{(12)\, h(d) \to e_1\})$

$\overset{\text{Orient }(E'.12)}{\underset{CC}{\Longrightarrow}} \; (\emptyset, R_5 \cup \{(13)\, e_1 \to e_4\})$

$\overset{\text{Collapse }(R.6, R.13)}{\underset{CC}{\Longrightarrow}} \; (\emptyset, (R_6 \setminus \{(6)\, f(a, e_1) \to e_3\}) \cup \{(14)\, f(a, e_4) \to e_3\})$

$\overset{\text{Deduce }(R.8, R.14)}{\underset{CC}{\Longrightarrow}} \; (\{(13)\, e_3 \approx e_5\}, R_7 \setminus \{(14)\, f(a, e_4) \to e_3\})$

$\overset{\text{Orient }(E'.13)}{\underset{CC}{\Longrightarrow}} \; (\emptyset, R_8 \cup \{(15)\, e_3 \to e_5\})$

$R = \{(3)\; f(e_4, a) \to e_5,\; (4)\; g(a) \to e_1,\; (5)\; g(b) \to e_2,$
$(7)\; h(d) \to e_4,\; (8)\; f(a, e_4) \to e_5,\; (9)\; e_2 \to e_4,$
$(10)\; c \to d,\; (11)\; f(b, e_4) \to e_3,\; (13)\; e_1 \to e_4$
$(15)\; e_3 \to e_5\}$

$f(b, g(b)) \not\approx f(h(c), a)$?

$f(b, g(b)) \xrightarrow{(5)}_R f(b, e_2) \xrightarrow{(9)}_R f(b, e_4) \xrightarrow{(11)}_R e_3 \xrightarrow{(15)}_R e_5$

$f(h(c), a) \xrightarrow{(10)}_R f(h(d), a) \xrightarrow{(7)}_R f(e_4, a) \xrightarrow{(3)}_R e_5$

$\Rightarrow f(b, g(b))\downarrow_R = f(h(c), a)\downarrow_R$

$\Rightarrow$ The conjunction is unsat.

Exercise 10.3

$$\underline{f(x_1,0)} \geq x_3 \wedge \underline{f(x_2,0)} \leq x_3 \wedge x_1 \approx x_2 \wedge x_3 - \underline{f(x_1,0)} \geq 1$$

Purified to

$$(x_5 \geq x_3) \wedge (x_6 \leq x_3) \wedge (x_1 \approx x_2) \wedge (x_3 - x_5 \geq 1) \wedge$$
$$(x_4 \approx 0) \wedge (x_5 \approx f(x_1, x_4)) \wedge x_6 \approx f(x_2, x_4)$$

$$N_1 = \{ \overset{①}{x_5 \geq x_3},\ x_6 \leq x_3,\ x_1 \approx x_2,\ \overset{②}{x_3 - x_5 \geq 1},\ x_4 \approx 0 \}$$

$$N_2 = \{ x_1 \approx x_2,\ x_5 \approx f(x_1, x_4),\ x_6 \approx f(x_2, x_4) \}$$

$(N_1 ; \emptyset ; N_2 ; \emptyset ; \bot)$

$\overset{\text{Fail}}{\underset{\text{NO}}{\Longrightarrow}} (N_1 ; \emptyset ; N_2 ; \emptyset ; \text{fail})$

$x_5 \overset{①}{\geq} x_3 \overset{②}{\geq} 1 + x_5 \models_{LA} \bot$

~~$\models x_5 \geq \ldots$~~

$\Rightarrow N_1 \models_{LA} \bot$

For N_2 we would have ~~gotten~~ derived:

$\{x_5 \approx x_6\}$