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Solutions informatik

Universität des Saarlandes
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Tutorials for "Automated Reasoning WS22/23"
Exercise sheet 6 Solutions

Exercise 6.1:
We have to check via CDCL(LRA) whether the clause set

$$\begin{array}{lll} N = \{ & (1) & 3x_1 + 4x_2 - 1 > 0, \\ & (2) & -x_1 + x_2 + 1 \geq 0, \\ & (3) & 2x_2 - x_3 \approx 0, \\ & (4) & x_3 - x_1 < -2 \vee x_2 > 1 \,\} \end{array}$$

is satisfiable where we consider the linear rational arithmetic theory.

First we have to define the bijection atr from the atoms in the theory to propositional variables and we define it as follows:

$N' = \mathrm{atr}^{-1}(N)$	$N = \mathrm{atr}(N')$
$3x_1 + 4x_2 - 1 > 0$	P_1
$-x_1 + x_2 + 1 \geq 0$	P_2
$2x_2 - x_3 \approx 0$	P_3
$x_3 - x_1 < -2$	P_4
$x_2 > 1$	P_5

and therefore we get the propositional clause set

$$N = \{(1)\, P_1, (2)\, P_2, (3)\, P_3, (4)\, P_4 \vee P_5\}.$$

Now we can apply $\Rightarrow_{\mathrm{CDCL(LRA)}}$ and get the following derivation:

$$\begin{array}{ll}
(\epsilon; N; \emptyset; 0; \top) & \\
\Rightarrow^{\text{Propagate(1)}}_{\text{CDCL(LRA)}} & (P_1^{(1)}; N; \emptyset; 0; \top) \\
\Rightarrow^{\text{Propagate(2)}}_{\text{CDCL(LRA)}} & (P_1^{(1)} P_2^{(2)}; N; \emptyset; 0; \top) \\
\Rightarrow^{\text{Propagate(3)}}_{\text{CDCL(LRA)}} & (P_1^{(1)} P_2^{(2)} P_3^{(3)}; N; \emptyset; 0; \top) \\
\Rightarrow^{\text{Decide}(P_4)}_{\text{CDCL(LRA)}} & (P_1^{(1)} P_2^{(2)} P_3^{(3)} P_4^{1}; N; \emptyset; 1; \top) \\
\Rightarrow^{\mathcal{T}\text{-Conflict}}_{\text{CDCL(LRA)}} & (\epsilon; N; \{(5)\, \neg P_1 \vee \neg P_2 \vee \neg P_3 \vee \neg P_4\}; 0; \top)
\end{array}$$

From the literals $\text{atr}^{-1}(P_3) = 2x_2 - x_3 \approx 0$ and $\text{atr}^{-1}(P_4) = x_3 - x_1 < -2$ follows $2x_2 < x_1 - 2$ and by isolating x_2 in the literal $\text{atr}^{-1}(P_2) = -x_1 + x_2 + 1 \geq 0$ we get $x_2 \geq x_1 - 1$ and therefore combining the two bounds for x_2 leads to $x_1 < 0$. Isolating x_2 in the literal $\text{atr}^{-1}(P_1) = 3x_1 + 4x_2 - 1 > 0$ leads to $4x_2 > -3x_1 + 1$ and combinig this with the bound $2x_2 < -2 + x_1$ leads to $x_1 > 1$. This contradicts $x_1 < 0$ and therefore $\mathcal{T}$-Conflict is applicable.

$$
\begin{aligned}
&(\epsilon; N; \{(5)\, \neg P_1 \vee \neg P_2 \vee \neg P_3 \vee \neg P_4\}; 0; \top)\\
\Rightarrow^{\text{Propagate},*}_{\text{CDCL(LRA)}} \quad &(P_1^{(1)} P_2^{(2)} P_3^{(3)}; N; U_5; 0; \top)\\
\Rightarrow^{\text{Propagate(5)}}_{\text{CDCL(LRA)}} \quad &(P_1^{(1)} P_2^{(2)} P_3^{(3)} \neg P_4^{(5)}; N; U_6; 0; \top)\\
\Rightarrow^{\text{Propagate(4)}}_{\text{CDCL(LRA)}} \quad &(P_1^{(1)} P_2^{(2)} P_3^{(3)} \neg P_4^{(5)} P_5^{(4)}; N; U_7; 0; \top)\\
\Rightarrow^{\mathcal{T}\text{-Success}}_{\text{CDCL(LRA)}} \quad &(P_1^{(1)} P_2^{(2)} P_3^{(3)} \neg P_4^{(5)} P_5^{(4)}; N; U_8; -1; \top)
\end{aligned}
$$

U_i is the set that we get after applying the i-th derivation step, so it is used to shorten the notation of our set of clauses after every derivation step. Note that after the application of $\mathcal{T}$-Conflict there are the same series of Propagation applications done as before.

Note that the literals P_1, P_2, P_3, $\neg P_4$ and P_5 satisfy the propostional system and $\mathcal{T}$-Success is applicable because $x_1 \mapsto 3$, $x_2 \mapsto 2$ and $x_3 \mapsto 4$ is a solution of the literals and therefore this variable assignment is a solution of the initial problem.

Exercise 6.2:

We have to check via CDCL(LRA) whether the clause set

$$
\begin{array}{lcc}
N = \{ & (1) & y < 5 + x \vee y > 5 + x,\\
 & (2) & x \approx z - 3,\\
 & (3) & y \leq 3x + 2 - z,\\
 & (4) & y - 11 + 3x \geq 2z,\\
 & (5) & y - z > 4\,\}
\end{array}
$$

is satisfiable.

First we have to define the bijection atr from the atoms in the theory to propositional variables and we define it as follows:

$N' = \text{atr}^{-1}(N)$	$N = \text{atr}(N')$
$y < 5 + x$	P_1
$y > 5 + x$	P_2
$x \approx z - 3$	P_3
$y \leq 3x + 2 - z$	P_4
$y - 11 + 3x \geq 2z$	P_5
$y - z > 4$	P_6

and therefore we get the propositional clause set

$$N = \{(1)\, P_1 \vee P_2, (2)\, P_3, (3)\, P_4, (4)\, P_5, (5)\, P_6\}.$$

Now we can apply $\Rightarrow_{\text{CDCL(LRA)}}$ and get the following derivation:

$$
\begin{aligned}
&(\epsilon; N; \emptyset; 0; \top)\\
\Rightarrow^{\text{Propagate(2)}}_{\text{CDCL(LRA)}} \quad & (P_3^{(2)}; N; \emptyset; 0; \top)\\
\Rightarrow^{\text{Propagate(3)}}_{\text{CDCL(LRA)}} \quad & (P_3^{(2)} P_4^{(3)}; N; \emptyset; 0; \top)\\
\Rightarrow^{\text{Propagate(4)}}_{\text{CDCL(LRA)}} \quad & (P_3^{(2)} P_4^{(3)} P_5^{(4)}; N; \emptyset; 0; \top)\\
\Rightarrow^{\text{Propagate(5)}}_{\text{CDCL(LRA)}} \quad & (P_3^{(2)} P_4^{(3)} P_5^{(4)} P_6^{(5)}; N; \emptyset; 0; \top)\\
\Rightarrow^{\text{Decide}(P_1)}_{\text{CDCL(LRA)}} \quad & (P_3^{(2)} P_4^{(3)} P_5^{(4)} P_6^{(5)} P_1^{1}; N; \emptyset; 1; \top)\\
\Rightarrow^{\mathcal{T}\text{-Conflict}}_{\text{CDCL(LRA)}} \quad & (\epsilon; N; \{(6)\, \neg P_1 \vee \neg P_3 \vee \neg P_6\}; 0; \top)
\end{aligned}
$$

We use $\text{atr}^{-1}(P_3) = x \approx z - 3$ to substitute x in $\text{atr}^{-1}(P_1) = y < 5 + x$ and get $y - z < 3$ but this contradicts $\text{atr}^{-1}(P_6) = y - z > 4$ and therefore $\mathcal{T}$-Conflict is applicable.

$$
\begin{aligned}
&(\epsilon; N; \{(6)\, \neg P_1 \vee \neg P_3 \vee \neg P_6\}; 0; \top)\\
\Rightarrow^{\text{Propagate},*}_{\text{CDCL(LRA)}} \quad & (P_3^{(2)} P_4^{(3)} P_5^{(4)} P_6^{(5)}; N; U_6; 0; \top)\\
\Rightarrow^{\text{Propagate(6)}}_{\text{CDCL(LRA)}} \quad & (P_3^{(2)} P_4^{(3)} P_5^{(4)} P_6^{(5)} \neg P_1^{(6)}; N; U_7; 0; \top)\\
\Rightarrow^{\text{Propagate(1)}}_{\text{CDCL(LRA)}} \quad & (P_3^{(2)} P_4^{(3)} P_5^{(4)} P_6^{(5)} \neg P_1^{(6)} P_2^{(1)}; N; U_8; 0; \top)\\
\Rightarrow^{\mathcal{T}\text{-Success(1)}}_{\text{CDCL(LRA)}} \quad & (P_3^{(2)} P_4^{(3)} P_5^{(4)} P_6^{(5)} \neg P_1^{(6)} P_2^{(1)}; N; U_9; -1; \top)
\end{aligned}
$$

U_i is the set that we get after applying the i-th derivation step, so it is used to shorten the notation of our set of clauses after every derivation step. Note that after the application of $\mathcal{T}$-Conflict there are the same series of Propagation applications done as before.

Note that the literals $\neg P_1$, P_2, P_3, P_4, P_5 and P_6 satisfy the propostional system and $\mathcal{T}$-Success is applicable because $x \mapsto 9$, $y \mapsto 17$ and $z \mapsto 12$ is a solution of the literals and therefore this variable assignment is a solution of the initial problem.

Exercise 6.3:

1.

$$
\begin{aligned}
& \{f(x, h(x,y)) = f(f(y,z), h(y,z'))\}\\
\Rightarrow^{\text{Decomposition}}_{\text{SU}} \quad & \{x = f(y,z),\;\; h(x,y) = h(y,z')\}\\
\Rightarrow^{\text{Decomposition}}_{\text{SU}} \quad & \{x = f(y,z),\;\; x = y,\;\; y = z'\}\\
\Rightarrow^{\text{Substitution}}_{\text{SU}} \quad & \{y = f(y,z),\;\; x = y,\;\; y = z'\}\\
\Rightarrow^{\text{Occurs Check}}_{\text{SU}} \quad & \bot
\end{aligned}
$$

$$
\begin{aligned}
& \{f(x, h(x,y)) = f(f(y,z), h(y,z'))\} \\
\Rightarrow^{\text{Decomposition}}_{\text{PU}} \; & \{x = f(y,z),\;\; h(x,y) = h(y,z')\} \\
\Rightarrow^{\text{Decomposition}}_{\text{PU}} \; & \{x = f(y,z),\;\; x = y,\;\; y = z'\} \\
\Rightarrow^{\text{Substitution}}_{\text{PU}} \; & \{y = f(y,z),\;\; x = y,\;\; y = z'\} \\
\Rightarrow^{\text{Occurs Check}}_{\text{PU}} \; & \perp
\end{aligned}
$$

2.

$$
\begin{aligned}
& \{h(x,y) = z,\;\; g(f(x,x)) = z',\;\; g(g(f(a,y))) = g(z')\} \\
\Rightarrow^{\text{Orient}}_{\text{SU}} \; & \{z = h(x,y),\;\; g(f(x,x)) = z',\;\; g(g(f(a,y))) = g(z')\} \\
\Rightarrow^{\text{Orient}}_{\text{SU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; g(g(f(a,y))) = g(z')\} \\
\Rightarrow^{\text{Decomposition}}_{\text{SU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; g(f(a,y)) = z'\} \\
\Rightarrow^{\text{Substitution}}_{\text{SU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; g(f(a,y)) = g(f(x,x))\} \\
\Rightarrow^{\text{Decomposition}}_{\text{SU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; f(a,y) = f(x,x)\} \\
\Rightarrow^{\text{Decomposition}}_{\text{SU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; a = x,\;\; y = x\} \\
\Rightarrow^{\text{Orient}}_{\text{SU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; x = a,\;\; y = x\} \\
\Rightarrow^{\text{Substitution}}_{\text{SU}} \; & \{z = h(a,y),\;\; z' = g(f(a,a)),\;\; x = a,\;\; y = a\} \\
\Rightarrow^{\text{Substitution}}_{\text{SU}} \; & \{z = h(a,a),\;\; z' = g(f(a,a)),\;\; x = a,\;\; y = a\}
\end{aligned}
$$

mgu: $\{z \mapsto h(a,a),\;\; z' \mapsto g(f(a,a)),\;\; x \mapsto a,\;\; y \mapsto a\}$

$$
\begin{aligned}
& \{h(x,y) = z,\;\; g(f(x,x)) = z',\;\; g(g(f(a,y))) = g(z')\} \\
\Rightarrow^{\text{Orient}}_{\text{PU}} \; & \{z = h(x,y),\;\; g(f(x,x)) = z',\;\; g(g(f(a,y))) = g(z')\} \\
\Rightarrow^{\text{Orient}}_{\text{PU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; g(g(f(a,y))) = g(z')\} \\
\Rightarrow^{\text{Decomposition}}_{\text{PU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; g(f(a,y)) = z'\} \\
\Rightarrow^{\text{Orient}}_{\text{PU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; z' = g(f(a,y))\} \\
\Rightarrow^{\text{Merge}}_{\text{PU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; g(f(x,x)) = g(f(a,y))\} \\
\Rightarrow^{\text{Decomposition}}_{\text{PU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; f(x,x) = f(a,y)\} \\
\Rightarrow^{\text{Decomposition}}_{\text{PU}} \; & \{z = h(x,y),\;\; z' = g(f(x,x)),\;\; x = a,\;\; x = y\} \\
\Rightarrow^{\text{Substitution}}_{\text{PU}} \; & \{z = h(y,y),\;\; z' = g(f(y,y)),\;\; y = a,\;\; x = y\}
\end{aligned}
$$

mgu: $\{x \mapsto y\}\{z \mapsto h(y,y)\}\{z' \mapsto g(f(y,y))\}\{y \mapsto a\}$
$= \{x \mapsto a,\;\; z \mapsto h(a,a),\;\; z' \mapsto g(f(a,a)),\;\; y \mapsto a\}$

3.

$$
\begin{aligned}
& \{h(x,y) = h(x',y'),\;\; y' = f(x,a),\;\; f(g(a),z) = y\} \\
\Rightarrow^{\text{Decomposition}}_{\text{SU}} \;& \{x = x',\;\; y = y',\;\; y' = f(x,a),\;\; f(g(a),z) = y\} \\
\Rightarrow^{\text{Substitution}}_{\text{SU}} \;& \{x = x',\;\; y = f(x,a),\;\; y' = f(x,a),\;\; f(g(a),z) = y\} \\
\Rightarrow^{\text{Substitution}}_{\text{SU}} \;& \{x = x',\;\; y = f(x,a),\;\; y' = f(x,a),\;\; f(g(a),z) = f(x,a)\} \\
\Rightarrow^{\text{Decomposition}}_{\text{SU}} \;& \{x = x',\;\; y = f(x,a),\;\; y' = f(x,a),\;\; g(a) = x,\;\; z = a\} \\
\Rightarrow^{\text{Orient}}_{\text{SU}} \;& \{x = x',\;\; y = f(x,a),\;\; y' = f(x,a),\;\; x = g(a),\;\; z = a\} \\
\Rightarrow^{\text{Substitution}}_{\text{SU}} \;& \{g(a) = x',\;\; y = f(g(a),a),\;\; y' = f(g(a),a),\;\; x = g(a), \\
& z = a\} \\
\Rightarrow^{\text{Orient}}_{\text{SU}} \;& \{x' = g(a),\;\; y = f(g(a),a),\;\; y' = f(g(a),a),\;\; x = g(a), \\
& z = a\}
\end{aligned}
$$

mgu: $\{x' \mapsto g(a),\;\; y \mapsto f(g(a),a),\;\; y' \mapsto f(g(a),a),\;\; x \mapsto g(a),\;\; z \mapsto a\}$

$$
\begin{aligned}
& \{h(x,y) = h(x',y'),\;\; y' = f(x,a),\;\; f(g(a),z) = y\} \\
\Rightarrow^{\text{Decomposition}}_{\text{PU}} \;& \{x = x',\;\; y = y',\;\; y' = f(x,a),\;\; f(g(a),z) = y\} \\
\Rightarrow^{\text{Substitution}}_{\text{PU}} \;& \{x = x',\;\; y = y',\;\; y' = f(x,a),\;\; f(g(a),z) = y'\} \\
\Rightarrow^{\text{Orient}}_{\text{PU}} \;& \{x = x',\;\; y = y',\;\; y' = f(x,a),\;\; y' = f(g(a),z)\} \\
\Rightarrow^{\text{Merge}}_{\text{PU}} \;& \{x = x',\;\; y = y',\;\; y' = f(x,a),\;\; f(x,a) = f(g(a),z)\} \\
\Rightarrow^{\text{Decomposition}}_{\text{PU}} \;& \{x = x',\;\; y = y',\;\; y' = f(x,a),\;\; x = g(a),\;\; a = z\} \\
\Rightarrow^{\text{Orient}}_{\text{PU}} \;& \{x = x',\;\; y = y',\;\; y' = f(x,a),\;\; x = g(a),\;\; z = a\} \\
\Rightarrow^{\text{Substitution}}_{\text{PU}} \;& \{x = x',\;\; y = y',\;\; y' = f(x',a),\;\; x' = g(a),\;\; z = a\}
\end{aligned}
$$

mgu: $\{z \mapsto a\}\{x \mapsto x'\}\{y \mapsto y'\}\{y' \mapsto f(x',a)\}\{x' \mapsto g(a)\}$
$= \{z \mapsto a,\;\; x \mapsto g(a),\;\; y \mapsto f(g(a),a),\;\; y' \mapsto f(g(a),a),\;\; x' \mapsto g(a)\}$

Exercise* 6.4:
For a state $E = \{s_1 = t_1, \ldots, s_n = t_n\}$ take the measure $\mu(E) := (M, u, v, k)$ where M is the multiset of term sizes of the side with larger term size, so $M = \{max\{|s_i|, |t_i|\} | 1 \leq i \leq n\}$. u is the number of unoriented equations $t = x$ in E where t is not a variable. v is the number of variable equations $x = y$ in E where $x \in vars(E')$ with $E = E' \uplus \{x = y\}$. k is the number of equations $x = t$ in E where there is another equation $(x = s) \in E$ where t and s are no variables. The state $\bot$ is mapped to $(\emptyset, 0, 0, 0)$. Then we need to show that the measure $\mu(E)$ decreases with each rule application $\Rightarrow_{\text{PU}}$ with respect to the lexicographic extension $\succ_{\text{lex}}$ of $>$ on the natural numbers and the multiset extension of $>$ on the natural numbers. After the rules Clash, Occurs Check and Cycle are applied $E = \bot$ and therefore the measure obviously decreases. The rule Tautology removes an equation and therefore decreases M and the measure. Decomposition replaces an equation with multiple equations where the term size is smaller than the term size of the original equation by at least 1. Therefore Decomposition

decreases M and also the measure. Applying Orient does not change M but decreases u by 1 and therefore also decreases the measure. The rule Substitution does not change M and u but decreases v by 1 and therefore also decreases the measure. Applying Merge does not change M because the right-hand side of the equation that is changed by Merge does not change and the term size of the right-hand side is larger than the term size of the left-hand side before and after the rule application. u and v are also not changed by Merge but k decreases and therefore also the measure decreases. So the measure $\mu(E)$ decreases with each rule application $\Rightarrow_{\mathrm{PU}}$ with respect to $\succ_{\mathrm{lex}}$. Therefore $\Rightarrow_{\mathrm{PU}}$ terminates.

It is not encouraged to prepare joint solutions, because we do not support joint exams.