# First-Order Logic Theories

## 3.17.1 Definition (First-Order Logic Theory)

Given a first-order many-sorted signature $\Sigma$, a *theory* $\mathcal{T}$ is a set of $\Sigma$-algebras.
For some first-order formula $\phi$ over $\Sigma$ we say that $\phi$ is *$\mathcal{T}$-satisfiable* if there is some $\mathcal{A} \in \mathcal{T}$ such that $\mathcal{A}(\beta) \models \phi$ for some $\beta$. We say that $\phi$ is *$\mathcal{T}$-valid* (*$\mathcal{T}$-unsatisfiable*) if for all $\mathcal{A} \in \mathcal{T}$ and all $\beta$ it holds $\mathcal{A}(\beta) \models \phi$ ($\mathcal{A}(\beta) \not\models \phi$). In case of validity I also write $\models_{\mathcal{T}} \phi$.

Alternatively, $\mathcal{T}$ may contain a set of satisfiable axioms which then stand for all algebras satisfying the axioms.

## 7.1.1 Definition (Convex Theory)

A theory $\mathcal{T}$ is *convex* if for a conjunction$\phi$ of literals with $\phi \models_{\mathcal{T}} x_1 \approx y_1 \vee \ldots \vee x_n \approx y_n$ then $\phi \models_{\mathcal{T}} x_k \approx y_k$ for some $k$.

# Nelson-Oppen Combination

## 7.1.2 Definition (Nelson-Oppen Basic Restrictions)

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two theories. Then the *Nelson-Oppen Basic Restrictions* are:

(i) There are decision procedures for for $\mathcal{T}_1$ and $\mathcal{T}_2$.

(ii) Each decision procedure returns a complete set of variable identities as consequence of a formula.

(iii) $\Sigma_1 \cap \Sigma_2 = \emptyset$ except for common sorts.

(iv) Both theories are convex.

(v) All domains of models in $\mathcal{T}_1$ and $\mathcal{T}_2$ are infinite.

Actually, restriction 7.1.2-2 is not needed, because a given finite quantifier-free formula $\phi$ over $\Sigma_1 \cup \Sigma_2$ contains only finitely many different variables. Now instead of putting the burden to identify variables on the decision procedure, all potential variable identifications can be guessed and tested afterwards. The disadvantage of this approach is, of course, that there are exponentially many identifications with respect to a fixed number of variables. Therefore, assuming 7.1.2-2 results in a more efficient procedure and is also supported by many procedures from Section 6.

Restriction 7.1.2-5 can be further relaxed to assume that the domains of all shared sorts of all models are either infinite or have the same number of elements.

# Purification

**Purify** $N \uplus \{L[t[s]_i]_p\} \Rightarrow_{\mathsf{NO}} N \uplus \{L[t[z]_i]_p, z \approx s\}$

if $t = f(t_1, \ldots, t_n)$, $s = h(s_1, \ldots, s_m)$, the function symbols $f$ and $h$ are from different signatures, $1 \leq i \leq n$, (i.e., $t_i = s$) and $z$ is a fresh variable of appropriate sort

# Nelson-Oppen Calculus

Now a Nelson-Oppen problem state is a five tuple $(N_1, E_1, N_2, E_2, s)$ with $s \in \{\top, \bot, \text{fail}\}$, the sets $E_1$ and $E_2$ contain variable equations, and $N_1$, $N_2$ literals over the respective signatures, where

| | |
|---|---|
| $(N_1; \emptyset; N_2; \emptyset; \bot)$ | is the start state for some purified set of atoms $N = N_1 \cup N_2$ where the $N_i$ are built from the respective signatures only |
| $(N_1; E_1; N_2; E_2; \text{fail})$ | is a final state, where $N_1 \cup N_2 \cup E_1 \cup E_2$ is unsatisfiable |
| $(N_1; E_1; N_2; E_2; \bot)$ | is an intermediate state, where $N_1 \cup E_2$ and $N_2 \cup E_1$ have to be checked for satisfiability |
| $(N_1; \emptyset; N_2; \emptyset; \top)$ | is a final state, where $N_1 \cup N_2$ is satisfiable |

**Solve** $(N_1; E_1; N_2; E_2; \bot) \Rightarrow_{\mathsf{NO}} (N_1'; E_1'; N_2'; E_2'; \bot)$
if $N_1' = N_1 \cup E_1 \cup E_2$ and $N_2' = N_2 \cup E_1 \cup E_2$ are both $\mathcal{T}_i$-satisfiable, respectively, $E_1'$ are all new variable equations derivable from $N_1'$, $E_2'$ are all new variable equations derivable from $N_2'$ and $E_1' \cup E_2' \neq \emptyset$

**Success** $(N_1; E_1; N_2; E_2; \bot) \Rightarrow_{\mathsf{NO}} (N_1'; \emptyset; N_2'; \emptyset; \top)$
if $N_1' = N_1 \cup E_1 \cup E_2$ and $N_2' = N_2 \cup E_1 \cup E_2$ are both $\mathcal{T}_i$-satisfiable, respectively, $E_1'$ are all new variable equations derivable from $N_1'$, $E_2'$ are all new variable equations derivable from $N_2'$ and $E_1' \cup E_2' = \emptyset$

**Fail** $(N_1; E_1; N_2; E_2; \bot) \Rightarrow_{\mathsf{NO}} (N_1; E_1; N_2; E_2; \mathsf{fail})$
if $N_1' = N_1 \cup E_1 \cup E_2$ or $N_2' = N_2 \cup E_1 \cup E_2$ is $\mathcal{T}_i$-unsatisfiable, respectively