Further Decidable FOL(T) Fragments

I assume in this section that the considered clause sets are sufficiently complete, but compactness needs not to hold. Furthermore, I don't consider equations, i.e., the SUP(T) calculus instantiates to the ordered resolution calculus modulo theories: Superposition Right only generates tautologies, Superposition Left becomes ordered resolution, Equality Factoring becomes factoring and Equality Resolution is not applicable.

Totally Ordered Clause Sets

For this fragment the only requirement is that satisfiability of constraints is decidable. For example, a constraint language of non-linear real arithmetic.

8.11.1 Definition (Closed Literal Set)

Let M be a set of first-order (non-equational) literals over Σ^F closed under SUP(T) inferences: for any two clauses $C_1, C_2 \in 2^M$ and SUP(T) inference D out of C_1, C_2, it holds $D \subset M$. Then M is called a *closed literal set*.

8.11.2 Definition (Totally Ordered Horn Clause Sets)

Let M be a closed literal set, and $\prec$ be a well-founded partial ordering on M stable under substitution and instantiation such that for all $\Lambda \parallel C \in N$: (i) $C \subset M$, (ii) C is Horn, (iii) if $C = C' \vee P(t_1, \ldots, t_n)$ then for all $L \in C'$: $L \prec P(t_1, \ldots, t_n)$. Then N is called a *totally ordered Horn clause set*.

Recall that an ordering $\prec$ is stable under substitution if $L \prec K$ implies $L\sigma \prec K\sigma$ for any σ. It is stable under instantiation if $L\sigma \preceq L$ for any σ.

8.11.3 Lemma (Saturation of Totally Ordered Horn Clause Sets Terminates)

Let N be a totally ordered Horn clause set. Then SUP(T) terminates on N.

8.11.4 Theorem (Satisfiability of Totally Ordered Horn Clause Sets is Decidable)

Let N be a totally ordered Horn clause set. Then satisfiability of N is decidable.

8.11.5 Example (Predicate Preference)

Let N be a clause set and $P_1, \dots, P_n$ be the predicates in N. Let $\prec$ be a total order on the P_i. It can be extended to literals by $P_i(t_1, \dots, t_n) \prec P_j(t_1, \dots, t_n)$ if $P_i \prec P_j$. The extension is stable under substitution and instantiation. Then satisfiability of any totally ordered Horn clause set with respect to $\prec$ is decidable.

Bernays-Schönfinkel with Simple Bounds

In this section I only consider clauses $\Lambda \parallel C$ where Λ is a conjunction of simple bounds over LRA and C is a Bernays-Schönfinkel clauses, i.e., the free part only consists of variables and constants. A *simple bound* is an (in)equality $x \# k$ where $k \in \mathbb{Z}$ and $\# \in \{<, \leq, >, \geq, =, \neq\}$. In Section 3.16 I have introduced a number of calculi that can decide the Bernays-Schönfinkel fragment. Here I prove that Bernays-Schönfinkel with simple bounds can also be decided by exactly the superposition variant introduced in Section 3.16.1. I assume that in any inferred clauses by superposition or instantiation the constraint is always simplified, i.e., for any clause $\Lambda \parallel C$ all constraint variables occur in C, for every such variable x there is at most one upper and one lower bound and duplicates are removed.

8.12.1 Lemma (BS with Simple Bounds Invariants)

Let N be a clause set of the Bernay-Schönfinkel fragment with simple bounds. Then

(1) Any inference between clauses from N results again in a BS clause with simple bounds. The class of Bernays-Schoenfinkel clauses with simple bounds is closed under SUP(T) inferences.

(2) Let $\{k_1, \ldots, k_n\}$ be all numeric values occurring in the constraints in N. Then also for any clause inferred by SUP(T) from N, only the numeric values $\{k_1, \ldots, k_n\}$ occur.

(3) For any arithmetic variable x at most n non-redundant simple bounds out of $\{k_1, \ldots, k_n\}$ can be generated.

Condensation-BS $(N \uplus \{\Lambda \parallel L_1 \vee \cdots \vee L_n\}) \Rightarrow_{\text{SUP}}$
$(N \cup \{\Lambda \parallel \text{rdup}((L_1 \vee \ldots L_n))\sigma_{i,j} \mid \sigma_{i,j} = \text{mgu}(L_i, L_j) \text{ and } \sigma_{i,j} \neq \bot\})$
provided any ground instantiation on the free variables $(L_1 \vee \cdots \vee L_n)\delta$ contains at least two duplicate literals with identical simple bounds

8.12.2 Lemma (BS with Simple Bounds Termination)

Let N be a BS clause set with simple bounds. There are only finitely many BSS clauses derivable from N where Condensation-BS is not applicable.

8.12.3 Theorem (BS with Simple Bounds Decidability)

Satisfiability of a BS clause set with simple bounds is decidable.

Bernays-Schönfinkel with Bounded Variables

In this section I only consider clauses $\Lambda \parallel C$ where for each arithmetic variable $x \in \text{vars}(C)$ there are bounds $i \leq x \leq j$ in Λ, $i, j \in \mathbb{Z}$, $i \leq j$ and all arithmetic variables are integer variables. The constraint λ may contain any further, even non-linear constraints. For example, the constraint language of non-linear integer arithmetic, i.e., polynomials are allowed.

8.13.1 Theorem (BS with Bounded Constraints is Decidable)

Satisfiability of BS clause sets with bounded constraints over the integers is decidable.