Given two formulas ϕ and ψ, ϕ *entails* ψ, or ψ is a *consequence* of ϕ, written $\phi \models \psi$, if for any algebra $\mathcal{A}$ and assignment β, if $\mathcal{A}, \beta \models \phi$ then $\mathcal{A}, \beta \models \psi$.

The formulas ϕ and ψ are called *equivalent*, written $\phi \models\!\!\!\!\dashv \psi$, if $\phi \models \psi$ and $\psi \models \phi$.

Two formulas ϕ and ψ are called *equisatisfiable*, if ϕ is satisfiable iff ψ is satisfiable (not necessarily in the same models).

The notions of "entailment", "equivalence" and "equisatisfiability" are naturally extended to sets of formulas, that are treated as conjunctions of single formulas.

Clauses are implicitly universally quantified disjunctions of literals. A clause C is satisfiable by an algebra $\mathcal{A}$ if for every assignment β there is a literal $L \in C$ with $\mathcal{A}, \beta \models L$.

Note that if $C = \{L_1, \ldots, L_k\}$ is a ground clause, i.e., every L_i is a ground literal, then $\mathcal{A} \models C$ if and only if there is a literal L_j in C so that $\mathcal{A} \models L_j$. A clause set N is satisfiable iff all clauses $C \in N$ are satisfiable by the same algebra $\mathcal{A}$. Accordingly, if N and M are two clause sets, $N \models M$ iff every model $\mathcal{A}$ of N is also a model of M.

3.3.1 Definition (Substitution (well-sorted))

A *well-sorted substitution* is a mapping $\sigma : \mathcal{X} \to T(\Sigma, \mathcal{X})$ so that

1. $\sigma(x) \neq x$ for only finitely many variables x and
2. $\mathrm{sort}(x) = \mathrm{sort}(\sigma(x))$ for every variable $x \in \mathcal{X}$.

The application $\sigma(x)$ of a substitution σ to a variable x is often written in postfix notation as $x\sigma$. The variable set $\mathrm{dom}(\sigma) := \{x \in \mathcal{X} \mid x\sigma \neq x\}$ is called the *domain* of σ.

The term set $\mathsf{codom}(\sigma) := \{x\sigma \mid x \in \mathsf{dom}(\sigma)\}$ is called the *codomain* of σ. From the above definition it follows that $\mathsf{dom}(\sigma)$ is finite for any substitution σ. The composition of two substitutions σ and τ is written as a juxtaposition $\sigma\tau$, i.e., $t\sigma\tau = (t\sigma)\tau$.

A substitution σ is called *idempotent* if $\sigma\sigma = \sigma$. A substitution σ is idempotent iff $\mathsf{dom}(\sigma) \cap \mathsf{vars}(\mathsf{codom}(\sigma)) = \emptyset$.

Substitutions are often written as sets of pairs $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ if $dom(\sigma) = \{x_1, \ldots, x_n\}$ and $x_i\sigma = t_i$ for every $i \in \{1, \ldots, n\}$.

The *modification* of a substitution σ at a variable x is defined as follows:

$$\sigma[x \mapsto t](y) = \begin{cases} t & \text{if } y = x \\ \sigma(y) & \text{otherwise} \end{cases}$$

A substitution σ is identified with its extension to formulas and defined as follows:

1. $\bot\sigma = \bot$,
2. $\top\sigma = \top$,
3. $(f(t_1, \ldots, t_n))\sigma = f(t_1\sigma, \ldots, t_n\sigma)$,
4. $(P(t_1, \ldots, t_n))\sigma = P(t_1\sigma, \ldots, t_n\sigma)$,
5. $(s \approx t)\sigma = (s\sigma \approx t\sigma)$,
6. $(\neg\phi)\sigma = \neg(\phi\sigma)$,
7. $(\phi \circ \psi)\sigma = \phi\sigma \circ \psi\sigma$ where $\circ \in \{\vee, \wedge\}$,
8. $(Qx\phi)\sigma = Qz(\phi\sigma[x \mapsto z])$ where $Q \in \{\forall, \exists\}$, z and x are of the same sort and z is a fresh variable.

The result $t\sigma$ ($\phi\sigma$) of applying a substitution σ to a term t (formula ϕ) is called an *instance* of t (ϕ).

The substitution σ is called *ground* if it maps every domain variable to a ground term, i.e., the codomain of σ consists of ground terms only.

If the application of a substitution σ to a term t (formula ϕ) produces a ground term $t\sigma$ (a variable-free formula, $\mathrm{vars}(\phi\sigma) = \emptyset$), then $t\sigma$ ($\phi\sigma$) is called *ground instance* of t (ϕ) and σ is called *grounding* for t (ϕ). The set of ground instances of a clause set N is given by $\mathrm{grd}(\Sigma, N) = \{C\sigma \mid C \in N, \sigma \text{ is grounding for } C\}$ is the set of *ground instances* of N.

A substitution σ is called a *variable renaming* if $\mathrm{codom}(\sigma) \subseteq \mathcal{X}$ and for any $x, y \in \mathcal{X}$, if $x \neq y$ then $x\sigma \neq y\sigma$.

3.3.2 Lemma (Substitutions and Assignments)

Let β be an assignment of some interpretation $\mathcal{A}$ of a term t and σ a substitution. Then

$$\beta(t\sigma) = \beta[x_1 \mapsto \beta(x_1\sigma), \ldots, x_n \mapsto \beta(x_n\sigma)](t)$$

where $\operatorname{dom}(\sigma) = \{x_1, \ldots, x_n\}$.

Firstly, we define the classic Herbrand interpretations for formulas without equality.

3.5.1 Definition (Herbrand Interpretation)

A *Herbrand Interpretation* (over Σ) is a Σ-algebra $\mathcal{H}$ such that

1. $S^{\mathcal{H}} := T_S(\Sigma)$ for every sort $S \in \mathcal{S}$
2. $f^{\mathcal{H}} : (s_1, \dots, s_n) \mapsto f(s_1, \dots, s_n)$ where $f \in \Omega$, $\operatorname{arity}(f) = n$, $s_i \in S_i^{\mathcal{H}}$ and $f : S_1 \times \dots \times S_n \to S$ is the sort declaration for f
3. $P^{\mathcal{H}} \subseteq (S_1^{\mathcal{H}} \times \dots \times S_m^{\mathcal{H}})$ where $P \in \Pi$, $\operatorname{arity}(P) = m$ and $P \subseteq S_1 \times \dots \times S_m$ is the sort declaration for P

3.5.2 Lemma (Herbrand Interpretations are Well-Defined)

Every Herbrand Interpretation is a Σ-algebra.

3.5.3 Proposition (Representing Herbrand Interpretations)

A Herbrand interpretation $\mathcal{A}$ can be uniquely determined by a set of ground atoms I

$$(s_1, \ldots, s_n) \in P^{\mathcal{A}} \text{ iff } P(s_1, \ldots, s_n) \in I$$

3.5.5 Theorem (Herbrand)

Let N be a finite set of Σ-clauses. Then N is satisfiable iff N has a Herbrand model over Σ iff $\mathrm{grd}(\Sigma, N)$ has a Herbrand model over Σ.

Orderings

Propositional superposition is based on an ordering on the propositional variables, Section 2.7. The ordering is total and well-founded. Basically, propositional variables correspond to ground atoms in first-order logic.

This section generalizes the ideas of the propositional superposition ordering to first-order logic. In first-order logic the ordering has to also consider terms and variables and operations on terms like the application of a substitution. See the first-order resolution calculus.

I first define the ordering on terms and then explain how it is extended to atoms.

3.11.1 Definition (Σ-Operation Compatible Relation)

A binary relation $\sqsupset$ over $T(\Sigma, \mathcal{X})$ is called *compatible with Σ-operations*, if $s \sqsupset s'$ implies

$$f(t_1, \dots, s, \dots, t_n) \sqsupset f(t_1, \dots, s', \dots, t_n)$$

for all $f \in \Omega$ and $s, s', t_i \in T(\Sigma, \mathcal{X})$.

3.11.2 Lemma (Σ-Operation Compatible Relation)

A relation $\sqsupset$ is compatible with Σ-operations iff $s \sqsupset s'$ implies $t[s]_p \sqsupset t[s']_p$ for all $s, s', t \in T(\Sigma, \mathcal{X})$ and $p \in pos(t)$.

3.11.3 Definition (Substitution Stable Relation, Rewrite Relation)

A binary relation $\sqsupset$ over $T(\Sigma, \mathcal{X})$ is called *stable under substitutions*, if $s \sqsupset s'$ implies $s\sigma \sqsupset s'\sigma$ for all $s, s' \in T(\Sigma, \mathcal{X})$ and substitutions σ.

A binary relation $\sqsupset$ is called a *rewrite relation*, if it is compatible with Σ-operations and stable under substitutions. A *rewrite ordering* is then an ordering that is a rewrite relation.

3.11.4 Definition (Subterm Ordering)

The *proper subterm ordering* $s > t$ is defined by $s > t$ iff $s|_p = t$ for some position $p \neq \epsilon$ of s.

3.11.5 Definition (Simplification Ordering)

A rewrite ordering $\succ$ over $T(\Sigma, \mathcal{X})$ is called *simplification ordering*, if it enjoys the *subterm property* $s \succ t$ implies $s > t$ for all $s, t \in T(\Sigma, \mathcal{X})$ of the same sort.

3.11.6 Definition (Lexicographical Path Ordering (LPO))

Let $\Sigma = (\mathcal{S}, \Omega, \Pi)$ be a signature and let $\succ$ be a strict partial ordering on operator symbols in Ω, called *precedence*. The *lexicographical path ordering* $\succ_{lpo}$ on $T(\Sigma, \mathcal{X})$ is defined as follows: if s, t are terms in $T_S(\Sigma, \mathcal{X})$ then $s \succ_{lpo} t$ iff

1. $t = x \in \mathcal{X}$, $x \in vars(s)$ and $s \neq t$ or
2. $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$ and
   - 2.1 $s_i \succeq_{lpo} t$ for some $i \in \{1, \ldots, n\}$ or
   - 2.2 $f \succ g$ and $s \succ_{lpo} t_j$ for every $j \in \{1, \ldots, m\}$ or
   - 2.3 $f = g$, $s \succ_{lpo} t_j$ for every $j \in \{1, \ldots, m\}$ and $(s_1, \ldots, s_n)(\succ_{lpo})_{lex}(t_1, \ldots, t_m)$.

3.11.7 Theorem (LPO Properties)

1. The LPO is a rewrite ordering.
2. LPO enjoys the subterm property, hence is a simplification ordering.
3. If the precedence $\succ$ is total on Ω then $\succ_{lpo}$ is total on the set of ground terms $T(\Sigma)$.
4. If Ω is finite then $\succ_{lpo}$ is well-founded.

3.11.9 Definition (The Knuth-Bendix Ordering)

Let $\Sigma = (\mathcal{S}, \Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω, let $w : \Omega \cup \mathcal{X} \to \mathbb{R}^+$ be a *weight function*, so that the following condition is satisfied:
$w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in \mathcal{X}$; $w(c) \geq w_0$ for all constants $c \in \Omega$.

Then, the weight function w can be extended to terms recursively:

$$w(f(t_1, \ldots, t_n)) = w(f) + \sum_{1 \leq i \leq n} w(t_i)$$

3.11.9 Definition (The Knuth-Bendix Ordering Ctd.)

or alternatively

$$\sum w(t) = \sum_{x \in vars(t)} w(x) \cdot \#(x,t) + \sum_{f \in \Omega} w(f) \cdot \#(f,t)$$

where $\#(a,t)$ is the number of occurrences of a in t.
The *Knuth-Bendix ordering* $\succ_{kbo}$ on $T(\Sigma, \mathcal{X})$ induced by $\succ$ and admissible w is defined by: $s \succ_{kbo} t$ iff

1 $\#(x,s) \geq \#(x,t)$ for all variables x and $w(s) > w(t)$, or
2 $\#(x,s) \geq \#(x,t)$ for all variables x, $w(s) = w(t)$, and
 (a) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f \succ g$, or
 (b) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and $(s_1, \ldots, s_m)(\succ_{kbo})_{lex}(t_1, \ldots, t_m)$.

3.11.10 Theorem (KBO Properties)

1. The KBO is a rewrite ordering.
2. KBO enjoys the subterm property, hence is a simplification ordering.
3. If the precedence $\succ$ is total on Ω then $\succ_{kbo}$ is total on the set of ground terms $T(\Sigma)$.
4. If Ω is finite then $\succ_{kbo}$ is well-founded.

The LPO ordering as well as the KBO ordering can be extended to atoms in a straightforward way. The precedence $\succ$ is extended to Π. For LPO atoms are then compared according to Definition 3.11.6-2. For KBO the weight function w is also extended to atoms by giving predicates a non-zero positive weight and then atoms are compared according to terms.

Actually, since atoms are never substituted for variables in first-order logic, an alternative to the above would be to first compare the predicate symbols and let $\succ$ decide the ordering. Only if the atoms share the same predicate symbol, the argument terms are considered, e.g., in a lexicographic way and are then compared with respect to KBO or LPO, respectively.