Rewrite Systems on Logics: Calculi

	Validity	Satisfiability
Sound	If the calculus derives a proof of validity for the formula, it is valid.	If the calculus derives satisfiability of the formula, it has a model.
Complete	If the formula is valid, a proof of validity is derivable by the calculus.	If the formula has a model, the calculus derives satisfiability.
Strongly Complete	For any validity proof of the formula, there is a derivation in the calculus producing this proof.	For any model of the formula, there is a derivation in the calculus producing this model.

Propositional Logic: Syntax

2.1.1 Definition (Propositional Formula)

The set PROP(Σ) of *propositional formulas* over a signature Σ, is inductively defined by:

PROP(Σ)	Comment
$\bot$	connective $\bot$ denotes "false"
$\top$	connective $\top$ denotes "true"
P	for any propositional variable $P \in \Sigma$
$(\neg\phi)$	connective $\neg$ denotes "negation"
$(\phi \land \psi)$	connective $\land$ denotes "conjunction"
$(\phi \lor \psi)$	connective $\lor$ denotes "disjunction"
$(\phi \rightarrow \psi)$	connective $\rightarrow$ denotes "implication"
$(\phi \leftrightarrow \psi)$	connective $\leftrightarrow$ denotes "equivalence"

where $\phi, \psi \in$ PROP(Σ).

Propositional Logic: Semantics

2.2.1 Definition ((Partial) Valuation)

A Σ-*valuation* is a map

$$\mathcal{A} : \Sigma \to \{0, 1\}.$$

where $\{0, 1\}$ is the set of *truth values*. A *partial* Σ-*valuation* is a map $\mathcal{A}' : \Sigma' \to \{0, 1\}$ where $\Sigma' \subseteq \Sigma$.

2.2.2 Definition (Semantics)

A Σ-valuation $\mathcal{A}$ is inductively extended from propositional variables to propositional formulas $\phi, \psi \in \text{PROP}(\Sigma)$ by

$$
\begin{array}{rcl}
\mathcal{A}(\bot) & := & 0 \\
\mathcal{A}(\top) & := & 1 \\
\mathcal{A}(\neg\phi) & := & 1 - \mathcal{A}(\phi) \\
\mathcal{A}(\phi \wedge \psi) & := & \min(\{\mathcal{A}(\phi), \mathcal{A}(\psi)\}) \\
\mathcal{A}(\phi \vee \psi) & := & \max(\{\mathcal{A}(\phi), \mathcal{A}(\psi)\}) \\
\mathcal{A}(\phi \rightarrow \psi) & := & \max(\{1 - \mathcal{A}(\phi), \mathcal{A}(\psi)\}) \\
\mathcal{A}(\phi \leftrightarrow \psi) & := & \text{if } \mathcal{A}(\phi) = \mathcal{A}(\psi) \text{ then 1 else 0}
\end{array}
$$

If $\mathcal{A}(\phi) = 1$ for some Σ-valuation $\mathcal{A}$ of a formula ϕ then ϕ is *satisfiable* and we write $\mathcal{A} \models \phi$. In this case $\mathcal{A}$ is a *model* of ϕ.

If $\mathcal{A}(\phi) = 1$ for all Σ-valuations $\mathcal{A}$ of a formula ϕ then ϕ is *valid* and we write $\models \phi$.

If there is no Σ-valuation $\mathcal{A}$ for a formula ϕ where $\mathcal{A}(\phi) = 1$ we say ϕ is *unsatisfiable*.

A formula ϕ *entails* ψ, written $\phi \models \psi$, if for all Σ-valuations $\mathcal{A}$ whenever $\mathcal{A} \models \phi$ then $\mathcal{A} \models \psi$.

Propositional Logic: Operations

2.1.2 Definition (Atom, Literal, Clause)

A propositional variable P is called an *atom*. It is also called a *(positive) literal* and its negation $\neg P$ is called a *(negative) literal*.

The functions comp and atom map a literal to its complement, or atom, respectively: if $\mathrm{comp}(\neg P) = P$ and $\mathrm{comp}(P) = \neg P$, $\mathrm{atom}(\neg P) = P$ and $\mathrm{atom}(P) = P$ for all $P \in \Sigma$. Literals are denoted by letters L, K. Two literals P and $\neg P$ are called *complementary*.

A disjunction of literals $L_1 \vee \ldots \vee L_n$ is called a *clause*. A clause is identified with the multiset of its literals.

2.1.3 Definition (Position)

A *position* is a word over $\mathbb{N}$. The set of positions of a formula ϕ is inductively defined by

$$
\begin{aligned}
\text{pos}(\phi) &:= \{\epsilon\} \text{ if } \phi \in \{\top, \bot\} \text{ or } \phi \in \Sigma \\
\text{pos}(\neg\phi) &:= \{\epsilon\} \cup \{1p \mid p \in \text{pos}(\phi)\} \\
\text{pos}(\phi \circ \psi) &:= \{\epsilon\} \cup \{1p \mid p \in pos(\phi)\} \cup \{2p \mid p \in \text{pos}(\psi)\}
\end{aligned}
$$

where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

The prefix order $\leq$ on positions is defined by $p \leq q$ if there is some p' such that $pp' = q$. Note that the prefix order is partial, e.g., the positions 12 and 21 are not comparable, they are "parallel", see below.

The relation $<$ is the strict part of $\leq$, i.e., $p < q$ if $p \leq q$ but not $q \leq p$.

The relation $\parallel$ denotes incomparable, also called parallel positions, i.e., $p \parallel q$ if neither $p \leq q$, nor $q \leq p$.

A position p is *above* q if $p \leq q$, p is *strictly above* q if $p < q$, and p and q are *parallel* if $p \parallel q$.

The *size* of a formula ϕ is given by the cardinality of $\text{pos}(\phi)$: $|\phi| := |\,\text{pos}(\phi)|$.

The *subformula* of ϕ at position $p \in \text{pos}(\phi)$ is inductively defined by $\phi|_\epsilon := \phi$, $\neg\phi|_{1p} := \phi|_p$, and $(\phi_1 \circ \phi_2)|_{ip} := \phi_i|_p$ where $i \in \{1, 2\}$, $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Finally, the *replacement* of a subformula at position $p \in \text{pos}(\phi)$ by a formula ψ is inductively defined by $\phi[\psi]_\epsilon := \psi$, $(\neg\phi)[\psi]_{1p} := \neg\phi[\psi]_p$, and $(\phi_1 \circ \phi_2)[\psi]_{1p} := (\phi_1[\psi]_p \circ \phi_2)$, $(\phi_1 \circ \phi_2)[\psi]_{2p} := (\phi_1 \circ \phi_2[\psi]_p)$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

2.1.5 Definition (Polarity)

The *polarity* of the subformula $\phi|_p$ of ϕ at position $p \in \mathrm{pos}(\phi)$ is inductively defined by

$$
\begin{array}{rcll}
\mathrm{pol}(\phi, \epsilon) & := & 1 & \\
\mathrm{pol}(\neg\phi, 1p) & := & -\mathrm{pol}(\phi, p) & \\
\mathrm{pol}(\phi_1 \circ \phi_2, ip) & := & \mathrm{pol}(\phi_i, p) & \text{if } \circ \in \{\wedge, \vee\}, i \in \{1, 2\} \\
\mathrm{pol}(\phi_1 \rightarrow \phi_2, 1p) & := & -\mathrm{pol}(\phi_1, p) & \\
\mathrm{pol}(\phi_1 \rightarrow \phi_2, 2p) & := & \mathrm{pol}(\phi_2, p) & \\
\mathrm{pol}(\phi_1 \leftrightarrow \phi_2, ip) & := & 0 & \text{if } i \in \{1, 2\}
\end{array}
$$

Valuations can be nicely represented by sets or sequences of literals that do not contain complementary literals nor duplicates.

If $\mathcal{A}$ is a (partial) valuation of domain Σ then it can be represented by the set
$\{P \mid P \in \Sigma \text{ and } \mathcal{A}(P) = 1\} \cup \{\neg P \mid P \in \Sigma \text{ and } \mathcal{A}(P) = 0\}$.

Another, equivalent representation are *Herbrand* interpretations that are sets of positive literals, where all atoms not contained in an Herbrand interpretation are false. If $\mathcal{A}$ is a total valuation of domain Σ then it corresponds to the Herbrand interpretation $\{P \mid P \in \Sigma \text{ and } \mathcal{A}(P) = 1\}$.

2.2.4 Theorem (Deduction Theorem)

$\phi \models \psi$ iff $\models \phi \rightarrow \psi$

2.2.6 Lemma (Formula Replacement)

Let ϕ be a propositional formula containing a subformula ψ at position p, i.e., $\phi|_p = \psi$. Furthermore, assume $\models \psi \leftrightarrow \chi$. Then $\models \phi \leftrightarrow \phi[\chi]_p$.

Propositional Tableau

2.4.1 Definition (α-, β-Formulas)

A formula ϕ is called an α-formula if ϕ is a formula $\neg\neg\phi_1$, $\phi_1 \wedge \phi_2$, $\phi_1 \leftrightarrow \phi_2$, $\neg(\phi_1 \vee \phi_2)$, or $\neg(\phi_1 \rightarrow \phi_2)$.

A formula ϕ is called a β-formula if ϕ is a formula $\phi_1 \vee \phi_2$, $\phi_1 \rightarrow \phi_2$, $\neg(\phi_1 \wedge \phi_2)$, or $\neg(\phi_1 \leftrightarrow \phi_2)$.

2.4.2 Definition (Direct Descendant)

Given an α- or β-formula ϕ, its direct descendants are as follows:

α	Left Descendant	Right Descendant
$\neg\neg\phi$	ϕ	ϕ
$\phi_1 \wedge \phi_2$	ϕ_1	ϕ_2
$\phi_1 \leftrightarrow \phi_2$	$\phi_1 \rightarrow \phi_2$	$\phi_2 \rightarrow \phi_1$
$\neg(\phi_1 \vee \phi_2)$	$\neg\phi_1$	$\neg\phi_2$
$\neg(\phi_1 \rightarrow \phi_2)$	ϕ_1	$\neg\phi_2$

β	Left Descendant	Right Descendant
$\phi_1 \vee \phi_2$	ϕ_1	ϕ_2
$\phi_1 \rightarrow \phi_2$	$\neg\phi_1$	ϕ_2
$\neg(\phi_1 \wedge \phi_2)$	$\neg\phi_1$	$\neg\phi_2$
$\neg(\phi_1 \leftrightarrow \phi_2)$	$\neg(\phi_1 \rightarrow \phi_2)$	$\neg(\phi_2 \rightarrow \phi_1)$

2.4.3 Proposition ()

For any valuation $\mathcal{A}$:

(i) if ϕ is an α-formula then $\mathcal{A}(\phi) = 1$ iff $\mathcal{A}(\phi_1) = 1$ and $\mathcal{A}(\phi_2) = 1$ for its descendants ϕ_1, ϕ_2.

(ii) if ϕ is a β-formula then $\mathcal{A}(\phi) = 1$ iff $\mathcal{A}(\phi_1) = 1$ or $\mathcal{A}(\phi_2) = 1$ for its descendants ϕ_1, ϕ_2.

Tableau Rewrite System

The tableau calculus operates on states that are sets of sequences of formulas. Semantically, the set represents a disjunction of sequences that are interpreted as conjunctions of the respective formulas.

A sequence of formulas $(\phi_1, \ldots, \phi_n)$ is called *closed* if there are two formulas ϕ_i and ϕ_j in the sequence where $\phi_i = \text{comp}(\phi_j)$.

A state is *closed* if all its formula sequences are closed.

The tableau calculus is a calculus showing unsatisfiability of a formula. Such calculi are called *refutational* calculi. Recall a formula ϕ is valid iff $\neg\phi$ is unsatisfiable.

A formula ϕ occurring in some sequence is called *open* if in case ϕ is an α-formula not both direct descendants are already part of the sequence and if it is a β-formula none of its descendants is part of the sequence.

Tableau Rewrite Rules

α-Expansion $N \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n)\} \Rightarrow_T N \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n, \psi_1, \psi_2)\}$
provided ψ is an open α-formula, ψ_1, ψ_2 its direct descendants and the sequence is not closed.

β-Expansion $N \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n)\} \Rightarrow_T N \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n, \psi_1)\} \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n, \psi_2)\}$
provided ψ is an open β-formula, ψ_1, ψ_2 its direct descendants and the sequence is not closed.