## 6.2.5 Lemma (Simplex State Invariants)

The following invariants hold for any state $(E_i; B_i; \beta_i; S_i; s_i)$ derived by $\Rightarrow_{\text{SIMP}}$ on a start state $(E_0; B_0; \beta_0; \emptyset; \top)$:

(i) for every dependent variable there is exactly one equation in $E$ defining the variable

(ii) dependent variables do not occur on the right hand side of an equation

(iii) $\text{LRA}(\beta) \models E_i$

(iv) for all independant variables $x$ either $\beta_i(x) = 0$ or $\beta_i(x) = c$ for some bound $x \circ c \in S_i$

(v) for all assignemnts $\alpha$ it holds $\text{LRA}(\alpha) \models E_0$ iff $\text{LRA}(\alpha) \models E_i$

### 6.2.6 Lemma (Simplex Run Invariants)

For any run of $\Rightarrow_{\text{SIMP}}$ from start state
$(E_0; B_0; \beta_0; \emptyset; \top) \Rightarrow_{\text{SIMP}} (E_1; B_1; \beta_1; S_1; s_1) \Rightarrow_{\text{SIMP}} \ldots$:

(i) the set $\{\beta_o, \beta_1, \ldots\}$ is finite

(ii) if the sets of dependent and independent variables for two equational systems $E_i$, $E_j$ coincide, then $E_i = E_j$

(iii) the set $\{E_o, E_1, \ldots\}$ is finite

(iv) let $S_i$ not contain contradictory bounds, then $(E_i; B_i; \beta_i; S_i; s_i) \Rightarrow_{\text{SIMP}}^{\text{FIV},*}$ is finite

## 6.2.7 Corollary (Infinite Runs Contain a Cycle)

Let $(E_0; B_0; \beta_0; \emptyset; \top) \Rightarrow_{\text{SIMP}} (E_1; B_1; \beta_1; S_1; s_1) \Rightarrow_{\text{SIMP}} \ldots$ be an infinite run. Then there are two states $(E_i; B_i; \beta_i; S_i; s_i)$, $(E_k; B_k; \beta_k; S_k; s_k)$ such that $i \neq k$ and $(E_i; B_i; \beta_i; S_i; s_i) = (E_k; B_k; \beta_k; S_k; s_k)$.

### 6.2.8 Definition (Reasonable Strategy)

A *reasonable* strategy prefers FailBounds over EstablishBounds and the FixDepVar rules select minimal variables $x$, $y$ in the ordering $\prec$.

## 6.2.9 Theorem (Simplex Soundness, Completeness & Termination)

Given a reasonable strategy and initial set $N$ of inequations and its separation into $E$ and $B$ :

(i) $\Rightarrow_{\text{SIMP}}$ terminates on $(E; B; \beta_0; \emptyset; \top)$,

(ii) if $(E; B; \beta_0; \emptyset; \top) \Rightarrow^*_{\text{SIMP}} (E'; B'; \beta; S; \bot)$ then $N$ has no solution,

(iii) if $(E; B; \beta_0; \emptyset; \top) \Rightarrow^*_{\text{SIMP}} (E'; \emptyset; \beta; B; \top)$ and $(E; \emptyset; \beta; B; \top)$ is a normal form, then $\text{LRA}(\beta) \models N$,

(iv) all final states $(E'; B'; \beta; S; s)$ match either (ii) or (iii).