Propositional Superposition

Propositional Superposition refines the propositional resolution calculus by

(i) ordering and selection restrictions on inferences,

(ii) an abstract redundancy notion,

(iii) the notion of a partial model, based on the ordering for inference guidance

(iv) a *saturation* concept.

Important: No implicit Condensation of literals!

2.7.1 Definition (Clause Ordering)

Let $\prec$ be a total strict ordering on Σ.

Then $\prec$ can be lifted to a total ordering on literals by $\prec \subseteq \prec_L$ and $P \prec_L \neg P$ and $\neg P \prec_L Q$, $\neg P \prec_L \neg Q$ for all $P \prec Q$.

The ordering $\prec_L$ can be lifted to a total ordering on clauses $\prec_C$ by considering the multiset extension of $\prec_L$ for clauses.

2.7.2 Proposition (Properties of the Clause Ordering)

(i) The orderings on literals and clauses are total and well-founded.
(ii) Let C and D be clauses with $P = \mathrm{atom}(\max(C))$, $Q = \mathrm{atom}(\max(D))$, where $\max(C)$ denotes the maximal literal in C.

(i) If $Q \prec_L P$ then $D \prec_C C$.
(ii) If $P = Q$, P occurs negatively in C but only positively in D, then $D \prec_C C$.

Eventually, I overload $\prec$ with $\prec_L$ and $\prec_C$.

For a clause set N, I define $N^{\prec C} = \{D \in N \mid D \prec C\}$.

Definition (Abstract Redundancy)

A clause C is *redundant* with respect to a clause set N if $N^{\prec C} \models C$.

2.7.5 Definition (Selection Function)

The selection function sel maps clauses to one of its negative literals or $\bot$.

If $\mathrm{sel}(C) = \neg P$ then $\neg P$ is called *selected* in C.

If $\mathrm{sel}(C) = \bot$ then no literal in C is *selected*.

2.7.6 Definition (Partial Model Construction)

Given a clause set N and an ordering $\prec$ we can construct a (partial) Herbrand model $N_\mathcal{I}$ for N inductively as follows:

$$
\begin{aligned}
N_C &:= \textstyle\bigcup_{D \prec C} \delta_D \\
\delta_D &:= \begin{cases} \{P\} & \text{if } D = D' \lor P,\ P \text{ strictly maximal, no literal} \\ & \text{selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases} \\
N_\mathcal{I} &:= \textstyle\bigcup_{C \in N} \delta_C
\end{aligned}
$$

Clauses C with $\delta_C \neq \emptyset$ are called *productive*.

2.7.7 Proposition (Model Construction Properties)

Some properties of the partial model construction.

(i) For every D with $(C \vee \neg P) \prec D$ we have $\delta_D \neq \{P\}$.

(ii) If $\delta_C = \{P\}$ then $N_C \cup \delta_C \models C$.

(iii) If $N_C \models D$ and $D \prec C$ then for all C' with $C \prec C'$ we have $N_{C'} \models D$ and in particular $N_{\mathcal{I}} \models D$.

(iv) There is no clause C with $P \vee P \prec C$ such that $\delta_C = \{P\}$.

Superposition Inference Rules

Superposition Left $(N \uplus \{C_1 \vee P, C_2 \vee \neg P\}) \Rightarrow_{\text{SUP}}$ $(N \cup \{C_1 \vee P, C_2 \vee \neg P\} \cup \{C_1 \vee C_2\})$

where (i) P is strictly maximal in $C_1 \vee P$ (ii) no literal in $C_1 \vee P$ is selected (iii) $\neg P$ is maximal and no literal selected in $C_2 \vee \neg P$, or $\neg P$ is selected in $C_2 \vee \neg P$

Factoring $(N \uplus \{C \vee P \vee P\}) \Rightarrow_{\text{SUP}}$ $(N \cup \{C \vee P \vee P\} \cup \{C \vee P\})$

where (i) P is maximal in $C \vee P \vee P$ (ii) no literal is selected in $C \vee P \vee P$

2.7.8 Definition (Saturation)

A set N of clauses is called *saturated up to redundancy*, if any inference from non-redundant clauses in N yields a redundant clause with respect to N or is already contained in N.