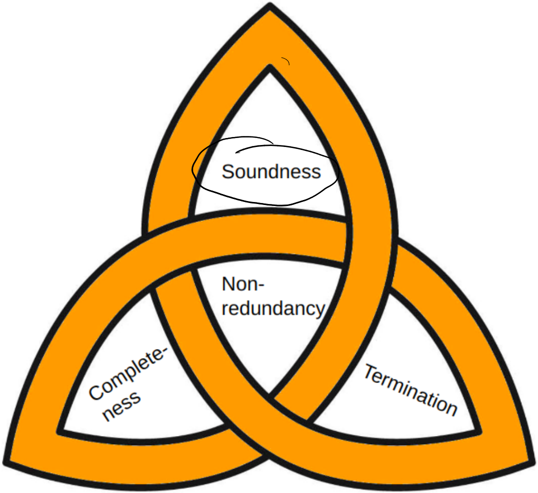
# SCL: Proofs



## 3.16.8 Definition (Sound States)

A state $(\Gamma; N; U; \beta; k; D)$ is *sound* if the following conditions hold:

1. $\Gamma$ is a consistent sequence of annotated ground literals, i.e. for a ground literal $L$ it cannot be that $L \in \Gamma$ and $\neg L \in \Gamma$
2. for each decomposition $\Gamma = \Gamma_1, L\sigma^{C \vee L \cdot \sigma}, \Gamma_2$ we have that $C\sigma$ is false under $\Gamma_1$ and $L\sigma$ is undefined under $\Gamma_1$, $N \cup U \models C \vee L$,
3. for each decomposition $\Gamma = \Gamma_1, L^k, \Gamma_2$ we have that $L$ is undefined in $\Gamma_1$,
4. $N \models U$,
5. if $D = C \cdot \sigma$ then $C\sigma$ is false under $\Gamma$ and $N \models C$. In particular, $\text{grd}^{\prec_\beta \beta}(N) \models C\sigma$,
6. for any $L \in \Gamma$ we have $L \prec_\beta \beta$ and there is a $C \in N \cup U$ such that $L \in C$.

### 3.16.9 Lemma (Soundness of the initial state)

The initial state $(\epsilon; N; \emptyset; \beta; 0; \top)$ is sound.

### Proof.

Criteria 1–3 and 6 are trivially satisfied by $\Gamma = \epsilon$. Furthermore, $N \models \emptyset$, fulfilling criterion 4. Lastly, criterion 5 is trivially fulfilled for $D = \top$. □

### 3.16.10 Theorem (Soundness of SCL)

All SCL rules preserve soundness, i.e. they map a sound state onto a sound state.

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**Corollary (Sondness of SCL)**

The rules of SCL are sound, hence SCL starting with an initial state is sound.

**Proof.**

Follows by induction over the size of the run. The base case is handled by Lemma 3.16.9, the induction step is contained in Theorem 3.16.10. □

### 3.16.12 Definition (Reasonable Runs)

A sequence of SCL rule applications is called a *reasonable run* if the rule Decide does not enable an immediate application of rule Conflict.

### 3.16.13 Definition (Regular Runs)

A sequence of SCL rule applications is called a *regular run* if it is a reasonable run and the rule Conflict has precedence over all other rules.

## 3.16.14 Theorem (Correct Termination)

If in a regular run no rules are applicable to a state $(\Gamma; N; U; \beta; k; D)$ then either $D = \bot$ and $N$ is unsatisfiable or $D = \top$ and $\mathrm{grd}(N)^{\prec_\beta \beta}$ is satisfiable and $\Gamma \models \mathrm{grd}(N)^{\prec_\beta \beta}$.

Proof Idea: Take a state where "no" rule is app.

- $(\Gamma; N; U; \beta; k; \top)$
  - undef. literal $<\beta$? $\rightarrow$ Decide, Propagate $\in C$ for $C \in (N \cup U)$
  - no undef. literal $<\beta$
    - $\Gamma \models \mathrm{grd}^{<\beta}(N)$: Done
    - $\Gamma \not\models \mathrm{grd}^{<\beta}(N)$: False clause under $\Gamma$ in $\mathrm{grd}^{<\beta}(N)$ $\rightarrow$ Choose as Conflict

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- $(\Gamma; N; U; \beta; k; C \cdot \sigma)$ (active conflict)
- $\Gamma = \varepsilon$: Soundness $\Gamma \models C\sigma \to C\sigma = \bot \to C = \bot$
  Soundness: $N \models C \to N \models \bot \to N$ unsat.
- $\Gamma = \Gamma', L$
  - $L$ is propagated:
    - comp$(L) \in C\sigma$: Resolve
    - otherwise: Skip
  - $L$ is a decision literal: apply either Backtrack, Factorize or Skip.

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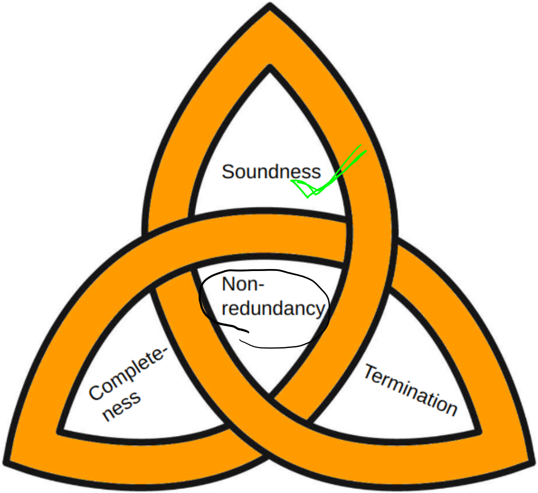

## 3.16.15 Lemma (Resolve in regular runs)

Consider the derivation of a conflict state $(\Gamma, L; N; U; \beta; k; \top) \Rightarrow_{\text{Conflict}} (\Gamma, L; N; U; \beta; k; D)$. In a regular run, during conflict resolution $L$ is not a decision literal and at least the literal $L$ is resolved.

Proof (idea).

How did we end up in $(\Gamma, L; N; U; \beta; k; \top)$

- Conflict, skip, Factorize, Resolve: obviously not ($D = \top$)
- Decide: not allowed by reasonability
- Backtrack: $(\Gamma, L, \Gamma'; N; U; k; D \cdot \sigma)$

  $\Rightarrow_{\text{Backtrack}} (\Gamma, L; N; U \cup \{D\}; k'; \top)$

  - conflict to $D$: impossible
  - conflict to any other clause in $N \cup U$? no! regularity

What's left? Propagate!

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What can we apply to $(\Gamma, L; N; U; \beta; k; D)$

- Backtrack: no
  requires $L$ to be a decision literal.
- Skip: no (if $L$ does not occur in $D$, apply conflict to $(\Gamma; N; U; \beta; k; \top)$) ↯ regularity.
- Factorize: "does not really make progress"
- Resolve ✓

## 3.16.16 Definition (State Induced Ordering)

Let $(L_1, L_2, \ldots, L_n; N; U; \beta; k; D)$ be a sound state of SCL. The trail induces a total well-founded strict order on the defined literals by

$$L_1 \prec_\Gamma \mathrm{comp}(L_1) \prec_\Gamma L_2 \prec_\Gamma \mathrm{comp}(L_2) \prec_\Gamma \cdots \prec_\Gamma L_n \prec_\Gamma \mathrm{comp}(L_n).$$

We extend $\prec_\Gamma$ to a strict total order on all literals where all undefined literals are larger than $\mathrm{comp}(L_n)$. We also extend $\prec_\Gamma$ to a strict total order on ground clauses by multiset extension and also on multisets of ground clauses and overload $\prec_\Gamma$ for all these cases. With $\preceq_\Gamma$ we denote the reflexive closure of $\prec_\Gamma$.

## 3.16.17 Theorem (Learned Clauses in Regular Runs)

Let $(\Gamma; N; U; \beta; k; C_0 \cdot \sigma_0)$ be the state resulting from the application of Conflict in a regular run and let $C$ be the clause learned at the end of the conflict resolution, then $C$ is not redundant with respect to $N \cup U$ and $\prec_\Gamma$.

(Idea) Consider $(\Gamma'; N; U; \beta; k; C \cdot \sigma) \Rightarrow$ Backtrack

- There was a literal $L$ in $C_0\sigma_0$ which is not in $C\sigma$ (3.16.15)
- $C\sigma$ is false under $\Gamma'$ (soundness)
- Assume $C\sigma$ is redundant

$\Gamma' \not\models \text{grd}(N \cup U)^{\leq_{\Gamma'}, C\sigma} \models C\sigma$, $\Gamma' \not\models C\sigma$

$\Rightarrow$ There is a false clause in $\text{grd}(N \cup U)^{\leq,}$
$\rightarrow$ we could have applied conflict earlier ↯ regularity

- During a run, the ordering of literals changes
- Hence, $\prec_\Gamma$ changes as well!
- Non-redundancy property of Theorem 3.16.17 reflects state at time of creation of learned clause
- At time of creation, no need to check for redundancy
- Still, **all** $\prec_\Gamma$ **contain** the fixed clause subset ordering $\prec_\subseteq$

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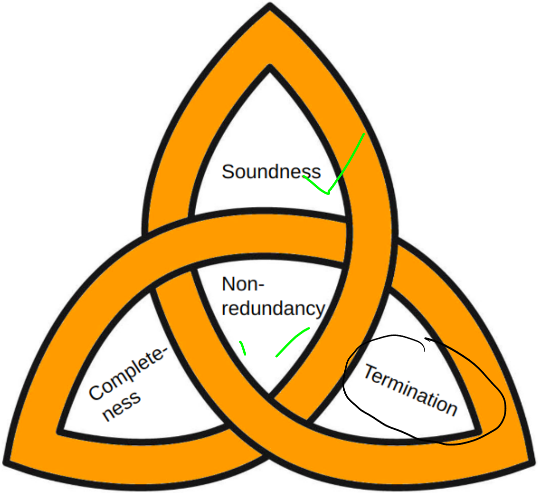

## 3.16.19 Theorem (Termination)

Any regular run of $\Rightarrow_{\mathrm{SCL}}$ terminates.

## Lemma (Termination without Backtrack)

Any regular run of $\Rightarrow_{\mathrm{SCL}}$ that does not use the Backtrack rule terminates.

$$\mathcal{M}(\Gamma, N; U; \beta; k; \top) = (1, \quad |\{P \mid P \prec_B \beta\}| - |\Gamma|, \quad 0 \quad)$$
$$\mathcal{M}(\Gamma, N; U; \beta; k; C) = (0, \quad \#\text{possible resolutions}, \quad |C| \quad)$$

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Decide, Propagate

$$\mathcal{M}(\Gamma, N; U; \beta; k; \top) = (1, \quad |\{P \mid P \prec_B \beta\}| - |\Gamma|, \quad 0 \quad )$$

$$\mathcal{M}(\Gamma, N; U; \beta; k; C) = (0, \quad \#\text{possible resolutions}, \quad |C| \quad )$$

conflict

Resolve, Skip

Factorize

## Lemma (Termination with Backtrack)

Any regular run of $\Rightarrow_{\mathrm{SCL}}$ cannot use the Backtrack rule infinitely often.

## Proof.

Firstly, for a regular run, by Theorem 3.16.17, all learned clauses are non-redundant under $\prec_\Gamma$. Those clauses are also non-redundant under the fixed subset ordering $\prec_\subseteq$, which is well-founded. Due to the restriction of all clauses to be smaller than $\{\beta\}$, the overall number of non-redundant ground clauses is finite. So Backtrack can only be invoked finitely many times. □

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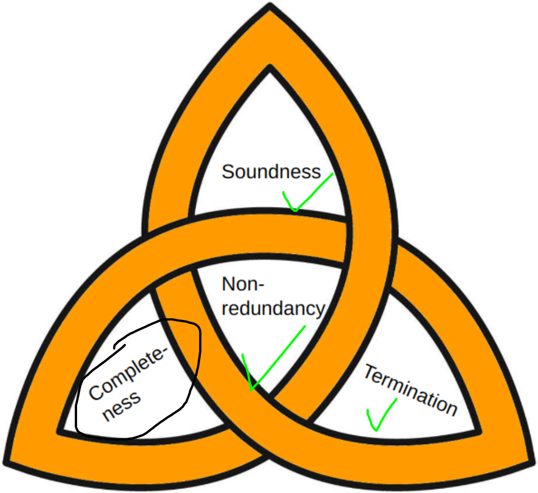

## 3.16.20 Theorem (SCL Refutational Completeness)

If $N$ is unsatisfiable, such that some finite $N' \subseteq \mathrm{grd}(N)$ is unsatisfiable and $\beta$ is $\prec_\beta$ larger than all literals in $N'$ then any regular run from $(\epsilon; N; \emptyset; \beta; 0; \top)$ of SCL derives $\bot$.

**Proof.**

By Theorem 3.16.19 and Theorem 3.16.14. □

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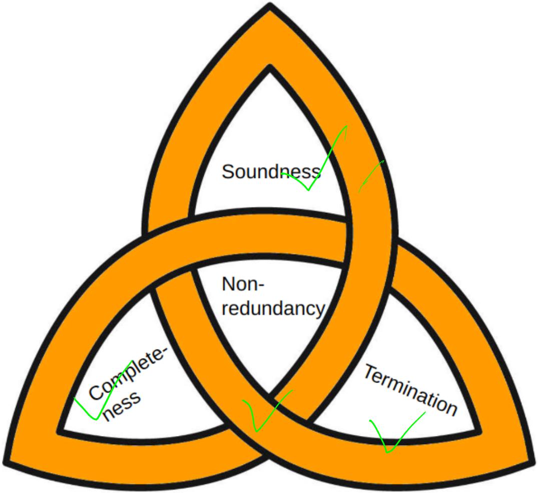

## 3.16.18 Theorem (BS Non-Redundancy is NEXPTIME-Complete)

Deciding non-redundancy of a BS clause $C$ with respect to a finite BS clause set $N^{\preceq C}$ is NEXPTIME-Complete.

- containment: solve $N^{\leq C} \models C$ ($C \in$ NEXPTIME)
- hardness: $N = \{C_1, \ldots, C_n\}$ finite BS-clause set

Define a $<_{LPO}$. Add fresh $P$ s.t. $P$ is $<_{LPO}$-larger than any literal in $N$.

in $\{C_1, \ldots, C_n, \{P\}\}$, $\{P\}$ is redundant $\iff$ $N$ unsatisfiable

(satisfiability of BS-clause is NEXPTIME-complete)

Obviously, given some unsatisfiable clause set $N$ there is no way to efiiciently compute some $\beta$ such that $\mathit{ground}(N)^{\prec_\beta}$ is unsatisfiable. Therefore, in an implementation, the below rule Grow is needed to eventually provide a semi-decision procedure.

**Grow** $\quad (\Gamma; N; U; \beta; k; \top) \; \Rightarrow_{\text{SCL}} \; (\epsilon; N; U; \beta'; 0; \top)$

provided $\Gamma \models \mathrm{grd}(N)^{\prec_\beta}$ and $\beta \prec_\beta \beta'$

## 3.16.21 Theorem (SCL decides the BS fragment)

SCL restricted to regular runs decides satisfiability of a BS clause set if $\beta$ is set appropriately.

## Proof.

Let $B$ be the set of constants in the BS clause set $N$. Then define $\prec_\beta$ and $\beta$ such that $L \prec_\beta \beta$ for all $L \in \mathrm{grd}^{\prec_\beta \beta}(N)$. Following the proof of Theorem 3.16.19, any SCL regular run will terminate on a BS clause set. □

# The End (of SCL)