Proof 3.12.9: by contradiction

$\bot \notin N$, C minimal from N with $N_I \not\models C$

(i) C is not redundant

by contradiction: assume C redundant

$C_1, \dots, C_n < C$ such that $C_1, \dots, C_n \models C$

$N_I \models C_1 \wedge \dots \wedge C_n$ but then $N_I \models C$ ↯ $N_I \not\models C$

(ii) C is not redundant

case analysis on the (strictly) max literal out of C

a) $P(t_1, \dots, t_n) \vee C' = C$ $\quad$ $P(t_1, \dots, t_n)$ is max but not strict max

$P(t_1, \dots, t_n) \vee P(t_1, \dots, t_n) \vee C'' = C$

$\leadsto$ Factoring $P(t_1, \dots, t_n) \vee C'' < C$ and ↯

$N_I \not\models P(t_1, \dots, t_n) \vee C''$ $\quad$ C was minimal

$P(t_1,\dots,t_n) \vee C'$ and $P(t_1,\dots,t_n)$ is strictly max

$N_I \not\models C' \rightsquigarrow N_C \not\models C' \rightsquigarrow \delta_C = \{P(t_1,\dots,t_n)\}$

$\lightning \quad N_I \not\models C$

b) $\neg P(t_1,\dots,t_n) \vee C' = C$ and is selected <u>or</u> maximal

there is a clause $D' \vee P(t_1,\dots,t_n)$ with $\delta_D = \{P(t_1,\dots,t_n)\}$

$D < C \qquad D$ is non redundant

Superposition left inference $\quad C' \vee D' < C$ ✓

because $\quad D < \{P(t_1,\dots,t_n)\}$

$N_I \not\models D'$ monotonicity of N_I

$N_I \not\models C' \quad \rightsquigarrow N_I \not\models C' \vee D' \quad \lightning$ C minimal

Semi-Decidability of FOL unsatisfiability

clause set N (with variables) is unsat

iff (Herbrand Theorem)

$gnd(N)$ is unsat

[ground all variables with ground terms from Ω]

$gnd(N)$ is infinite

iff (Sup Completeness)

$\bot$ can be derived by Sup from $gnd(N)$

BS clause set

$$N = \{\ P(x,y) \lor Q(a)$$

$$\neg P(b,z) \lor \neg Q(z) \qquad \Omega = \{a,b\}$$

$$\neg P(x,x)\ \}$$

$$\mathrm{gnd}_{\Sigma}(N) = \{$$

(1) $P(a,a) \lor Q(a)$

(2) $P(a,b) \lor Q(a)$

(3) $P(b,a) \lor Q(a)$

(4) $P(b,b) \lor Q(a)$

(5) $\neg P(b,a) \lor \neg Q(a)$

(6) $\neg P(b,b) \lor \neg Q(b)$

(7) $\neg P(a,a)$

(8) $\neg P(b,b)$ $\}$

$\sim$ sat{ { 7 SR 1 (9) $Q(a)$ subsumes (1), …, (4)

9 SR 5 : (10) $\neg P(b,a)$

(7) $\neg P(a,a)$

(8) $\neg P(b,b)$ }

$$N_I = \{Q(a)\}$$

Full FOL

$$N = \left\{ \frac{\neg P(x) \lor P(f(x))}{P(a)} \right\} \qquad \Sigma = \{a, f\}$$

$$\text{LPO}: f > P > a$$

$$gnd_\Sigma(N) = \{ \neg P(a) \lor P(f(a)),\ P(a),\ \neg P(f(a)) \lor P(f(f(a))),\ \neg P(f(f(a))) \lor P(f(f(f(a)))),\ \dots \}$$

- a) select always negative literals
  sat $\{ P(a), P(f(a)), P(f(f(a))), \dots \} = gnd(N)_I$
- b) KBO with $w(\_) = 1$, no selection
  no inferences $gnd(N)_I = \{ P(a), P(f(a)), \dots \}$

Craig's theorem: (Prop Style)

$\Phi \models \psi \rightsquigarrow \Phi \models \chi$ and $\chi \models \psi$

χ only contains prop variables occurring **both** in Φ and ψ

without select $\chi = \psi$

recall: prove $\Phi \models \psi$
refute $\Phi \wedge \neg\psi$

Proof: by Superposition

transform Φ and ψ into cnf: $N = cnf(\Phi)$
$M = cnf(\neg\psi)$

choose ordering variables occurring in Φ but **not** in ψ are max

saturate N into N^* by Sup

then saturate $N^* \cup M$, no inferences inside N^*,

$\rightsquigarrow$ derive $\bot$ in this proof there are only clauses from N^* that build out of symbols for M

χ conjunction of all clauses from N^* used to derive $\bot$ from $N^* \cup M$

$\equiv$ Example: $\Phi = (Q \wedge P \to R)$

$\Psi = (P \to R \vee S)$

common symbols P, R

$\chi = P \to R$

S only occurs in Ψ

Q only occurs in Φ

$N = \mathrm{cnf}(\Phi) = \{Q, \neg P \vee R\}$ Ordering $Q > P > R$

$M = \mathrm{cnf}(\neg\Psi) = \{P, \neg R, \neg S\}$

$\mathrm{sat}(N) \rightsquigarrow$ no inference $\rightsquigarrow N^* = \{Q, \neg P^* \vee R\}$

$\mathrm{sat}(N^* \cup M) \rightsquigarrow \mathrm{sup}(P^*, \neg P^* \vee R) = R \quad \mathrm{sup}(R, \neg R) = \bot$