$x > y, x = y + 1$ || $P(x, y) \vee \neg Q(a)$ LRA

$(x > y \wedge x = y + 1) \rightarrow (P(x, y) \vee \neg Q(a))$ $\mathbb{Q}$

$x \not> y \vee x \neq y + 1 \vee P(x, y) \vee \neg Q(a)$

$P(x, y)$ x is free?

x of sort Rational

$\Longrightarrow$ many-sorted

Löwenheim-Skolem (192x):

If a first-order sentence is valid over an infinite domain, it is valid over any infinite Domain

$U = \mathbb{N}, \mathbb{Q}, \mathbb{R}$ $\Rightarrow$ no, You cannot define the naturals in FOL

LRA ⇝ Domain $\mathbb{Q}$

⇝ we can define $\mathbb{N}$

$a \neq 0$
$a \neq 1$
$a \neq 2$
⋮

Löw. Compactness

$\mathbb{Q} \uplus \{a\}$

FOL (LRA) (no free symbols: you cannot $\mathbb{N}$)

↪ $\forall x \exists y \, (x + y \geq 1 \vee 3x - y = 6)$

$=$

→ BS (LRA)

functions + skolem

x, y of sort Rational

$Nat(0)$

$y = x + 1 \;\|\; \neg Nat(x) \vee Nat(y)$

$0 < x < 1 \;\|\; \neg Nat(x)$

$x < 0 \;\|\; \neg Nat(x)$

$y + 1 = x, x > 1 \;\|\; \neg Nat(x) \vee Nat(y)$

$\mathbb{N} \subseteq Nat^{I}$

$\mathbb{N} = Nat^{I}$

$\exists^{*} x \; \Phi$

$\| \neg P(a) \quad \| P(0)$

$\| Nat(a) \quad y = x + 1 \,\| P(x) \vee P(y) \quad \forall x \, \Phi \rightsquigarrow \exists x \, \Phi \rightsquigarrow x \to a$ a sort Rat

$R(\underline{x+z}, y)$ not abstracted

$z = x+2 \parallel R(z, y)$ abstracted

$P(0)$ not abstracted

$x = 0 \parallel P(x)$ abstracted

$P(0)$

$x \neq 0 \lor P(x)$

$P(1+n)$

$\neg P(2)$

$\parallel$

$x = 1+n \parallel P(x)$

$y = 2 \parallel \neg P(y)$

Res

$2 = 1+n \parallel \bot$

Contradiction

LRA

$x > 2 \;\|\; P(x)$

$x' < 3 \;\|\; \neg P(x')$

$\overset{\text{Res}}{\Longrightarrow}$ $\{x \to x'\}$

$2 < x' < 3 \;\|\; \bot$

$\neg(2 < x' < 3) \vee \bot$

$\neg(2 < x' \wedge x' < 3)$

$\forall x' (2 \geq x' \vee x' \geq 3)$ false in LRA

$\exists x' (2 < x' \wedge x' < 3)$ valid true