First-Order Superposition

Now the result for ground superposition are lifted to superposition on first-order clauses with variables, still without equality.

The completeness proof of ground superposition above talks about (strictly) maximal literals of ground clauses. The non-ground calculus considers those literals that correspond to (strictly) maximal literals of ground instances.

The used ordering is exactly the ordering of Definition 3.12.1 where clauses with variables are projected to their ground instances for ordering computations.

3.13.1 Definition (Maximal Literal)

A literal L is called *maximal* in a clause C if and only if there exists a grounding substitution σ so that $L\sigma$ is maximal in $C\sigma$, i.e., there is no different $L' \in C$: $L\sigma \prec L'\sigma$. The literal L is called *strictly maximal* if there is no different $L' \in C$ such that $L\sigma \preceq L'\sigma$.

Note that the orderings KBO and LPO cannot be total on atoms with variables, because they are stable under substitutions. Therefore, maximality can also be defined on the basis of absence of greater literals. A literal L is called *maximal* in a clause C if $L \nprec L'$ for all other literals $L' \in C$. It is called *strictly maximal* in a clause C if $L \npreceq L'$ for all other literals $L' \in C$.

Superposition Left

$(N \uplus \{C_1 \lor P(t_1,\ldots,t_n), C_2 \lor \neg P(s_1,\ldots,s_n)\}) \;\Rightarrow_{\text{SUP}}$
$(N \cup \{C_1 \lor P(t_1,\ldots,t_n), C_2 \lor \neg P(s_1,\ldots,s_n)\} \cup \{(C_1 \lor C_2)\sigma\})$

where (i) $P(t_1,\ldots,t_n)\sigma$ is strictly maximal in $(C_1 \lor P(t_1,\ldots,t_n))\sigma$ (ii) no literal in $C_1 \lor P(t_1,\ldots,t_n)$ is selected (iii) $\neg P(s_1,\ldots,s_n)\sigma$ is maximal and no literal selected in $(C_2 \lor \neg P(s_1,\ldots,s_n))\sigma$, or $\neg P(s_1,\ldots,s_n)$ is selected in $(C_2 \lor \neg P(s_1,\ldots,s_n))\sigma$ (iv) σ is the mgu of $P(t_1,\ldots,t_n)$ and $P(s_1,\ldots,s_n)$

Factoring

$(N \uplus \{C \lor P(t_1,\ldots,t_n) \lor P(s_1,\ldots,s_n)\}) \;\Rightarrow_{\text{SUP}}$
$(N \cup \{C \lor P(t_1,\ldots,t_n) \lor P(s_1,\ldots,s_n)\} \cup \{(C \lor P(t_1,\ldots,t_n))\sigma\})$

where (i) $P(t_1,\ldots,t_n)\sigma$ is maximal in $(C \lor P(t_1,\ldots,t_n) \lor P(s_1,\ldots,s_n))\sigma$ (ii) no literal is selected in $C \lor P(t_1,\ldots,t_n) \lor P(s_1,\ldots,s_n)$ (iii) σ is the mgu of $P(t_1,\ldots,t_n)$ and $P(s_1,\ldots,s_n)$

Note that the above inference rules Superposition Left and Factoring are generalizations of their respective counterparts from the ground superposition calculus above. Therefore, on ground clauses they coincide. Therefore, we can safely overload them in the sequel.

3.13.3 Definition (Abstract Redundancy)

A clause C is *redundant* with respect to a clause set N if for all ground instances $C\sigma$ there are clauses $\{C_1, \ldots, C_n\} \subseteq N$ with ground instances $C_1\tau_1, \ldots, C_n\tau_n$ such that $C_i\tau_i \prec C\sigma$ for all i and $C_1\tau_1, \ldots, C_n\tau_n \models C\sigma$.

3.13.4 Definition (Saturation)

A set N of clauses is called *saturated up to redundancy*, if any inference from non-redundant clauses in N yields a redundant clause with respect to N or is contained in N.

In contrast to the ground case, the above abstract notion of redundancy is not effective, i.e., it is undecidable for some clause C whether it is redundant, in general. Nevertheless, the concrete ground redundancy notions carry over to the non-ground case. Note also that a clause C is contained in N modulo renaming of variables.

Subsumption $(N \uplus \{C_1, C_2\}) \Rightarrow_{\text{SUP}} (N \cup \{C_1\})$

provided $C_1\sigma \subset C_2$ for some σ

Tautology Deletion $(N \uplus \{C \vee P(t_1, \ldots, t_n) \vee \neg P(t_1, \ldots, t_n)\})$
$\Rightarrow_{\text{SUP}} (N)$

Let rdup be a function from clauses to clauses that removes duplicate literals, i.e., $\mathrm{rdup}(C) = C'$ where $C' \subseteq C$, C' does not contain any duplicate literals, and for each $L \in C$ also $L \in C'$.

Condensation $(N \uplus \{C_1 \lor L \lor L'\}) \Rightarrow_{\text{SUP}}$ $(N \cup \{\mathrm{rdup}((C_1 \lor L \lor L')\sigma)\})$
provided $L\sigma = L'$ and $\mathrm{rdup}((C_1 \lor L \lor L')\sigma)$ subsumes $C_1 \lor L \lor L'$ for some σ

Subsumption Resolution $(N \uplus \{C_1 \lor L, C_2 \lor L'\}) \Rightarrow_{\text{SUP}}$ $(N \cup \{C_1 \lor L, C_2\})$
where $L\sigma = \neg L'$ and $C_1\sigma \subseteq C_2$ for some σ

3.13.7 Lemma (Lifting)

Let $D \vee L$ and $C \vee L'$ be variable-disjoint clauses and σ a grounding substitution for $C \vee L$ and $D \vee L'$. If there is a superposition left inference
$(N \uplus \{(D \vee L)\sigma, (C \vee L')\sigma\}) \Rightarrow_{\text{SUP}}$
$(N \cup \{(D \vee L)\sigma, (C \vee L')\sigma\} \cup \{D\sigma \vee C\sigma\})$ and if
$\text{sel}((D \vee L)\sigma) = \text{sel}((D \vee L))\sigma$, $\text{sel}((C \vee L')\sigma) = \text{sel}((C \vee L'))\sigma$, then there exists a mgu τ such that
$(N \uplus \{D \vee L, C \vee L'\}) \Rightarrow_{\text{SUP}} (N \cup \{D \vee L, C \vee L'\} \cup \{(D \vee C)\tau\})$.

Let $C \vee L \vee L'$ be a clause and σ a grounding substitution for $C \vee L \vee L'$. If there is a factoring inference
$(N \uplus \{(C \vee L \vee L')\sigma\}) \Rightarrow_{\text{SUP}} (N \cup \{(C \vee L \vee L')\sigma\} \cup \{(C \vee L)\sigma\})$
and if $\text{sel}((C \vee L \vee L')\sigma) = \text{sel}((C \vee L \vee L'))\sigma$, then there exists a mgu τ such that
$(N \uplus \{C \vee L \vee L'\}) \Rightarrow_{\text{SUP}} (N \cup \{C \vee L \vee L'\} \cup \{(C \vee L)\tau\})$

3.13.8 Example (First-Order Reductions are not Liftable)

Consider the two clauses $P(x) \lor Q(x)$, $P(g(y))$ and grounding substitution $\{x \mapsto g(a), y \mapsto a\}$. Then $P(g(y))\sigma$ subsumes $(P(x) \lor Q(x))\sigma$ but $P(g(y))$ does not subsume $P(x) \lor Q(x)$. For all other reduction rules similar examples can be constructed.

3.13.9 Lemma (Soundness and Completeness)

First-Order Superposition is sound and complete.

3.13.10 Lemma (Redundant Clauses are Obsolete)

If a clause set N is unsatisfiable, then there is a derivation $N \Rightarrow^*_{\text{SUP}} N'$ such that $\bot \in N'$ and no clause in the derivation of $\bot$ is redundant.

3.13.11 Lemma (Model Property)

If N is a saturated clause set and $\bot \notin N$ then $\mathrm{grd}(\Sigma, N)_{\mathcal{I}} \models N$.

Decision Procedures for BS

3.15.3 Definition (Bernays-Schoenfinkel Fragment (BS))

A formula of the Bernays-Schoenfinkel fragment has the form $\exists\vec{x}.\forall\vec{y}.\phi$ such that ϕ does not contain quantifiers nor non-constant function symbols.

3.15.4 Theorem (BS is decidable)

Unsatisfiability of a BS clause set is decidable.

$$1 : \neg R(x, y) \vee \neg R(y, z) \vee R(x, z)$$

$$2 : R(x, y) \vee R(y, x)$$

A state is now a set of clause sets. Let k be the number of different constants $a_1, \ldots, a_k$ in the initial clause set N. Then the initial state is the set $M = \{N\}$, Superposition Left is adopted to the new setting, Factoring is no longer needed and the rules Instantiate and Split are added. The variables $x_1, \ldots, x_k$ constitute a *variable chain* between literals L_1, L_k inside a clause C, if there are literals $\{L_1, \ldots, L_k\} \subseteq C$ such that $x_i \in (\text{vars}(L_i) \cap \text{vars}(L_{i+1}))$, $1 \leq i < k$.

Superposition BS
$M \uplus \{N \uplus \{P(t_1,\ldots,t_n), C \vee \neg P(s_1,\ldots,s_n)\}\} \Rightarrow_{\text{SUPBS}}$
$M \cup \{N \cup \{P(t_1,\ldots,t_n), C \vee \neg P(s_1,\ldots,s_n)\} \cup \{C\sigma\}\}$
where (i) $\neg P(s_1,\ldots,s_n)$ is selected in $(C \vee \neg P(s_1,\ldots,s_n))\sigma$ (ii) σ is the mgu of $P(t_1,\ldots,t_n)$ and $P(s_1,\ldots,s_n)$
(iii) $C \vee \neg P(s_1,\ldots,s_n)$ is a Horn clause

Instantiation $M \uplus \{N \uplus \{C \vee A_1 \vee A_2\}\} \Rightarrow_{\text{SUPBS}}$
$M \cup \{N \cup \{(C \vee A_1 \vee A_2)\sigma_i \mid \sigma_i = \{x \mapsto a_i\}, 1 \leq i \leq k\}\}$
where x occurs in a variable chain between A_1 and A_2

Split $M \uplus \{N \uplus \{C_1 \vee A_1 \vee C_2 \vee A_2\}\}$
$\Rightarrow_{\text{SUPBS}}$ $M \cup \{N \cup \{C_1 \vee A_1\}, N \cup \{C_2 \vee A_2\}\}$
where $\text{vars}(C_1 \vee A_1) \cap \text{vars}(C_2 \vee A_2) = \emptyset$

3.16.1 Definition (Rigorous Selection Strategy)

A selection strategy is *rigorous* of in any clause containing a negative literal, a negative literal is selected.

3.16.2 Lemma (SUPBS Basic Properties)

The SUPBS rules have the following properties:

1. Superposition BS is sound.
2. Instantiation is sound and complete.
3. Split is sound and complete.

Alternative Condensation Rule

The Condensation-BS rule turns Superposition (Resolution) into a decision procedure for the Bernays-Schönfinkel fragment and is an alternative to the SUPBS calculus.

Condensation-BS $(N \uplus \{L_1 \lor \cdots \lor L_n\}) \Rightarrow_{\text{SUP}}$
$(N \cup \{\text{rdup}((L_1 \lor \ldots L_n)\sigma_{i,j}) \mid \sigma_{i,j} = \text{mgu}(L_i, L_j)$ and $\sigma_{i,j} \neq \bot\})$
provided any ground instance $(L_1 \lor \cdots \lor L_n)\delta$ contains at least two duplicate literals