Simplex

The Simplex algorithm is the prime algorithm for solving optimization problems of systems of linear inequations over the rationals. For automated reasoning optimization at the level of conjunctions of inequations is not in focus. Rather, solvability of a set of linear inequations as a subproblem of some theory combination is the typical application. In this context the simplex algorithm is useful as well, due to its incremental nature. If an inequation $t \circ c$, $\circ \in \{\leq, \geq, <, >\}$, $t = \sum a_i x_i$, $a_i, c \in \mathbb{Q}$, is added to a set N of inequations where the simplex algorithm has already found a solution for N, the algorithm needs not to start from scratch. Instead it continues with the solution found for N. In practice, it turns out that then typically only few steps are needed to derive a solution for $N \cup \{t \circ d\}$ if it exists.

Firstly, the problem is rescritcted to non-strict inequations. Starting point is a set N (conjunction) of (non-strict) inequations of the form $(\sum_{x_j \in X} a_{i,j} x_j) \circ_i c_i$ where $\circ_i \in \{\geq, \leq\}$ for all i. Note that an equation $\sum a_i x_i = c$ can be encoded by two inequations $\{\sum a_i x_i \leq c, \sum a_i x_i \geq c\}$.

The variables occurring in N are assumed to be totally ordered by some ordering $\prec$. The ordering $\prec$ will eventualy guarantee termination of the simplex algorithm, see Definition 6.2.10 and Theorem 6.2.11 below. I assume the x_j to be all different, without loss of generality $x_j \prec x_{j+1}$, and I assume that all coefficients are normalized by the gcd of the $a_{i,j}$ for all j: if the gcd is different from 1 for one inequation, it is used for division of all coefficients of the inequation.

The goal is to decide whether there exists an assignment β from the x_j into $\mathbb{Q}$ such that

$$\mathrm{LRA}(\beta) \models \bigwedge_i [(\sum_{x_j \in X} a_{i,j} x_j) \circ_i c_i]$$

or equivalently, $\mathrm{LRA}(\beta) \models N$. So the x_j are free variables, i.e., placeholders for concrete values, i.e., existentially quantified.

The first step is to transform the set N of inequations into two disjoint sets E, B of equations and simple bounds, respectively. The set E contains equations of the form $y_i \approx \sum_{x_j \in X} a_{i,j} x_j$, where the y_i are fresh and the set B contains the respective simple bounds $y_i \circ_i c_i$. In case the original inequation from N was already a simple bound, i.e., of the form $x_j \circ_j c_j$ it is simply moved to B. If in N left hand sides of ineqations $(\sum_{x_j \in X} a_{i,j} x_j) \circ_i c_i$ are shared, it is sufficient to introduce one equation for the respective left hand side. The y_i are also part of the total ordering $\prec$ on all variables.

The two representations are equivalent:

$$\mathrm{LRA}(\beta) \models N$$

iff

$$\mathrm{LRA}(\beta[y_i \mapsto \beta(\textstyle\sum_{x_j \in X} a_{i,j} x_j)]) \models E$$

and

$$\mathrm{LRA}(\beta[y_i \mapsto \beta(\textstyle\sum_{x_j \in X} a_{i,j} x_j)]) \models B.$$

Given E and B a variable z is called *dependent* if it occurs on the left hand side of an equation in E, i.e., there is an equation $(z \approx \sum_{x_j \in X} a_{i,j} x_j) \in E$, and in case such a defining equation for z does not exist in E the variable z is called *independent*. Note that by construction the initial y_i are all dependent and do not occur on the right hand side of an equation.

Given a dependant variable x, an independent variable y, and a set of equations E, the *pivot* operation exchanges the roles of x, y in E where y occurs with non-zero coefficient in the defining equation of x. Let $(x \approx ay + t) \in E$ be the defining equation of x in E. When writing $(x \approx ay + t)$ for some equation, I always assume that $y \notin \text{vars}(t)$. Let E' be E without the defining equation of x. Then

$$\text{piv}(E, x, y) := \{y \approx \frac{1}{a}x + \frac{1}{-a}t\} \cup E'\{y \mapsto (\frac{1}{a}x + \frac{1}{-a}t)\}$$

Given an assignment β, an independent variable y, a rational value c, and a set of equations E then the *update* of β with respect to y, c, and E is

$$\mathsf{upd}(\beta, y, c, E) := \beta[y \mapsto c, \{x \mapsto \beta[y \mapsto c](t) \mid x \approx t \in E\}]$$

A Simplex problem state is a quintuple $(E; B; \beta; S; s)$ where E is a set of equations; B a set of simple bounds; β an assignment to all variables in E, B; S a set of derived bounds, and s the status of the problem with $s \in \{\top, \mathrm{IV}, \mathrm{DV}, \bot\}$. The state $s = \top$ indicates that $\mathrm{LRA}(\beta) \models S$; the state $s = \mathrm{IV}$ that potentially $\mathrm{LRA}(\beta) \not\models x \circ c$ for some independent variable x, $x \circ c \in S$; the state $s = \mathrm{DV}$ that $\mathrm{LRA}(\beta) \models x \circ c$ for all independent variables x, $x \circ c \in S$, but potentially $\mathrm{LRA}(\beta) \not\models x' \circ c'$ for some dependent variable x', $x' \circ c' \in S$; and the state $s = \bot$ that the problem is unsatisfiable.

The following states can be distinguished:

$(E; B; \beta_0; \emptyset; \top)$	is the start state for N and its transformation into E, B, and assignment $\beta_0(x) := 0$ for all $x \in \text{vars}(E \cup B)$
$(E; \emptyset; \beta; S; \top)$	is a final state, where $\text{LRA}(\beta) \models E \cup S$ and hence the problem is solvable
$(E; B; \beta; S; \bot)$	is a final state, where $E \cup B \cup S$ has no model

The important invariants of the simplex rules are:

(i) for every dependent variable there is exactly one equation in E defining the variable and

(ii) dependent variables do not occur on the right hand side of an equation,

(iii) $\mathrm{LRA}(\beta) \models E$

These invariants are maintained by a pivot (piv) or an update (upd) operation.

EstablishBound $(E; B \uplus \{x \circ c\}; \beta; S; \top) \Rightarrow_{\text{SIMP}}$
$(E; B; \beta; S \cup \{x \circ c\}; \text{IV})$

AckBounds $(E; B; \beta; S; s) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \top)$
if $\text{LRA}(\beta) \models S$, $s \in \{\text{IV}, \text{DV}\}$

FixIndepVar $(E; B; \beta; S; \text{IV}) \Rightarrow_{\text{SIMP}}$
$(E; B; \text{upd}(\beta, x, c, E); S; \text{IV})$
if $(x \circ c) \in S$, $\text{LRA}(\beta) \not\models x \circ c$, x independent

AckIndepBound $\quad (E; B; \beta; S; \mathrm{IV}) \Rightarrow_{\mathrm{SIMP}} (E; B; \beta; S; \mathrm{DV})$

if $\mathrm{LRA}(\beta) \models x \circ c$, for all independent variables x with bounds $x \circ c$ in S

FixDepVar$\leq (E; B; \beta; S; \mathrm{DV}) \Rightarrow_{\mathrm{SIMP}} (E'; B; \mathrm{upd}(\beta, x, c, E'); S; \mathrm{DV})$

if $(x \leq c) \in S$, x dependent, $\mathrm{LRA}(\beta) \not\models x \leq c$, there is an independent variable y and equation $(x \approx ay + t) \in E$ where $(a < 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a > 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$ and $E' := \mathrm{piv}(E, x, y)$

FixDepVar$\geq (E; B; \beta; S; \mathrm{DV}) \Rightarrow_{\mathrm{SIMP}} (E'; B; \mathrm{upd}(\beta, x, c, E'); S; \mathrm{DV})$

if $(x \geq c) \in S$, x dependent, $\mathrm{LRA}(\beta) \not\models x \geq c$, there is an independent variable y and equation $(x \approx ay + t) \in E$ where $(a > 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a < 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$ and $E' := \mathrm{piv}(E, x, y)$

FailBounds $(E; B; \beta; S; \top) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \bot)$
if there are two contradicting bounds $x \leq c_1$ and $x \geq c_2$ in $B \cup S$ for some variable x

FailDepVar$\leq$ $(E; B; \beta; S; \text{DV}) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \bot)$
if $(x \leq c) \in S$, x dependent, $\text{LRA}(\beta) \not\models x \leq c$ and there is no independent variable y and equation $(x \approx ay + t) \in E$ where ($a < 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S$) or ($a > 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S$)

FailDepVar$\geq$ $(E; B; \beta; S; \text{DV}) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \bot)$
if $(x \geq c) \in S$, x dependent, $\beta \not\models_{\text{LA}} x \geq c$ and there is no independent variable y and equation $(x \approx ay + t) \in E$ where (if $a > 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S$) or (if $a < 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S$)

6.2.7 Lemma (Simplex State Invariants)

The following invariants hold for any state $(E_i; B_i; \beta_i; S_i; s_i)$ derived by $\Rightarrow_{\mathrm{SIMP}}$ on a start state $(E_0; B_0; \beta_0; \emptyset; \top)$:

(i) for every dependent variable there is exactly one equation in E defining the variable

(ii) dependent variables do not occur on the right hand side of an equation

(iii) $\mathrm{LRA}(\beta) \models E_i$

(iv) for all independant variables x either $\beta_i(x) = 0$ or $\beta_i(x) = c$ for some bound $x \circ c \in S_i$

(v) for all assignemnts α it holds $\mathrm{LRA}(\alpha) \models E_0$ iff $\mathrm{LRA}(\alpha) \models E_i$

6.2.8 Lemma (Simplex Run Invariants)

For any run of $\Rightarrow_{\text{SIMP}}$ from start state
$(E_0; B_0; \beta_0; \emptyset; \top) \Rightarrow_{\text{SIMP}} (E_1; B_1; \beta_1; S_1; s_1) \Rightarrow_{\text{SIMP}} \ldots$:

(i) the set $\{\beta_o, \beta_1, \ldots\}$ is finite

(ii) if the sets of dependent and independent variables for two equational systems E_i, E_j coincide, then $E_i = E_j$

(iii) the set $\{E_o, E_1, \ldots\}$ is finite

(iv) let S_i not contain contradictory bounds, then $(E_i; B_i; \beta_i; S_i; s_i) \Rightarrow^{\text{FIV},*}_{\text{SIMP}}$ is finite

6.2.9 Corollary (Infinite Runs Contain a Cycle)

Let $(E_0; B_0; \beta_0; \emptyset; \top) \Rightarrow_{\text{SIMP}} (E_1; B_1; \beta_1; S_1; s_1) \Rightarrow_{\text{SIMP}} \ldots$ be an infinite run. Then there are two states $(E_i; B_i; \beta_i; S_i; s_i)$, $(E_k; B_k; \beta_k; S_k; s_k)$ such that $i \neq k$ and $(E_i; B_i; \beta_i; S_i; s_i) = (E_k; B_k; \beta_k; S_k; s_k)$.

6.2.10 Definition (Reasonable Strategy)

A *reasonable* strategy prefers FailBounds over EstablishBounds and the FixDepVar rules select minimal variables x, y in the ordering $\prec$.

6.2.11 Theorem (Simplex Soundness, Completeness & Termination)

Given a reasonable strategy and initial set N of inequations and its separation into E and B :

(i) $\Rightarrow_{\text{SIMP}}$ terminates on $(E; B; \beta_0; \emptyset; \top)$,

(ii) if $(E; B; \beta_0; \emptyset; \top) \Rightarrow^*_{\text{SIMP}} (E'; B'; \beta; S; \bot)$ then N has no solution,

(iii) if $(E; B; \beta_0; \emptyset; \top) \Rightarrow^*_{\text{SIMP}} (E'; \emptyset; \beta; B; \top)$ and $(E; \emptyset; \beta; B; \top)$ is a normal form, then $\text{LRA}(\beta) \models N$,

(iv) all final states $(E'; B'; \beta; S; s)$ match either (ii) or (iii).