3.12 First-Order Ground Superposition

Propositional clauses and ground clauses are essentially the same, as long as equational atoms are not considered. This section deals only with ground clauses and recalls mostly the material from Section 2.7 for first-order ground clauses. The main difference is that the atom ordering is more complicated, see Section 3.11. Let N be a possibly infinite set of ground clauses.

Definition 3.12.1 (Ground Clause Ordering)**.** Let $\prec$ be a strict rewrite ordering total on ground terms and ground atoms. Then $\prec$ can be lifted to a total ordering $\prec_L$ on literals by its multiset extension $\prec_{\text{mul}}$ where a positive literal

$P(t_1, \ldots, t_n)$ is mapped to the multiset $\{P(t_1, \ldots, t_n)\}$ and a negative literal $\neg P(t_1, \ldots, t_n)$ to the multiset $\{P(t_1, \ldots, t_n), P(t_1, \ldots, t_n)\}$. The ordering $\prec_L$ is further lifted to a total ordering on clauses $\prec_C$ by considering the multiset extension of $\prec_L$ for clauses.

Proposition 3.12.2 (Properties of the Ground Clause Ordering)**.**

1. The orderings on literals and clauses are total and well-founded.

2. Let C and D be clauses with $P(t_1, \ldots, t_n) = \operatorname{atom}(\max(C))$, $Q(s_1, \ldots, s_m) = \operatorname{atom}(\max(D))$, where $\max(C)$ denotes the maximal literal in C.

   (a) If $Q(s_1, \ldots, s_m) \prec_L P(t_1, \ldots, t_n)$ then $D \prec_C C$.

   (b) If $P(t_1, \ldots, t_n) = Q(s_1, \ldots, s_m)$, $P(t_1, \ldots, t_n)$ occurs negatively in C but only positively in D, then $D \prec_C C$.

Eventually, as I did for propositional logic, I overload $\prec$ with $\prec_L$ and $\prec_C$. So if $\prec$ is applied to literals it denotes $\prec_L$, if it is applied to clauses, it denotes $\prec_C$. Note that $\prec$ is a total ordering on literals and clauses as well. For superposition, inferences are restricted to maximal literals with respect to $\prec$. For a clause set N, I define $N^{\prec C} = \{D \in N \mid D \prec C\}$.

Definition 3.12.3 (Abstract Redundancy)**.** A ground clause C is *redundant* with respect to a set of ground clauses N if $N^{\prec C} \models C$.

Tautologies are redundant. Subsumed clauses are redundant if $\subseteq$ is strict. Duplicate clauses are anyway eliminated quietly because the calculus operates on sets of clauses.

Note that for finite N, and any $C \in N$ redundancy $N^{\prec C} \models C$ can be decided but is as hard as testing unsatisfiability for a clause set N. So the goal is to invent redundancy notions that can be efficiently decided and that are useful.

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Definition 3.12.4 (Selection Function)**.** The selection function sel maps clauses to one of its negative literals or $\bot$. If $\operatorname{sel}(C) = \neg P(t_1, \ldots, t_n)$ then $\neg P(t_1, \ldots, t_n)$ is called *selected* in C. If $\operatorname{sel}(C) = \bot$ then no literal in C is *selected*.

The selection function is, in addition to the ordering, a further means to restrict superposition inferences. If a negative literal is selected in a clause, any superposition inference must be on the selected literal.

Definition 3.12.5 (Partial Model Construction)**.** Given a clause set N, an ordering $\prec$, and a selection function sel the (partial) model $N_{\mathcal{I}}$ for N is inductively constructed as follows:

$$
\begin{array}{lcl}
N_C & := & \bigcup_{D \prec C} \delta_D \\[1em]
\delta_D & := & \begin{cases} \{P(t_1,\ldots,t_n)\} & \text{if } D = D' \lor P(t_1,\ldots,t_n), P(t_1,\ldots,t_n) \text{ strictly} \\ & \text{maximal, } \operatorname{sel}(D) = \bot \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases} \\
N_{\mathcal{I}} & := & \bigcup_{C \in N} \delta_C
\end{array}
$$

Clauses C with $\delta_C \neq \emptyset$ are called *productive*.

Proposition 3.12.6 (Properties of the Model Operator)**.** Some properties of the partial model construction.

1. For every D with $(C \lor \neg P(t_1,\ldots,t_n)) \prec D$ we have $\delta_D \neq \{P(t_1,\ldots,t_n)\}$.
2. If $\delta_C = \{P(t_1,\ldots,t_n)\}$ then $N_C \cup \delta_C \models C$.
3. If $N_C \models D$ and $D \prec C$ then for all C' with $C \prec C'$ we have $N_{C'} \models D$ and in particular $N_{\mathcal{I}} \models D$.
4. There is no clause C with $P(t_1,\ldots,t_n) \lor P(t_1,\ldots,t_n) \prec C$ such that $\delta_C = \{P(t_1,\ldots,t_n)\}$.

Please properly distinguish: N is a set of clauses interpreted as the conjunction of all clauses. $N^{\prec C}$ is of set of clauses from N strictly smaller than C with respect to $\prec$. $N_{\mathcal{I}}$, N_C are Herbrand interpretations (see Proposition 3.5.3). $N_{\mathcal{I}}$ is the overall (partial) model for N, whereas N_C is generated from all clauses from N strictly smaller than C.

Superposition Left $(N \uplus \{C_1 \lor P(t_1,\ldots,t_n), C_2 \lor \neg P(t_1,\ldots,t_n)\}) \Rightarrow_{\text{SUP}}$ $(N \cup \{C_1 \lor P(t_1,\ldots,t_n), C_2 \lor \neg P(t_1,\ldots,t_n)\} \cup \{C_1 \lor C_2\})$

where (i) $P(t_1,\ldots,t_n)$ is strictly maximal in $C_1 \lor P(t_1,\ldots,t_n)$ (ii) no literal in $C_1 \lor P(t_1,\ldots,t_n)$ is selected (iii) $\neg P(t_1,\ldots,t_n)$ is maximal and no literal selected in $C_2 \lor \neg P(t_1,\ldots,t_n)$, or $\neg P(t_1,\ldots,t_n)$ is selected in $C_2 \lor \neg P(t_1,\ldots,t_n)$

Factoring $(N \uplus \{C \lor P(t_1,\ldots,t_n) \lor P(t_1,\ldots,t_n)\}) \Rightarrow_{\text{SUP}}$ $(N \cup \{C \lor P(t_1,\ldots,t_n) \lor P(t_1,\ldots,t_n)\} \cup \{C \lor P(t_1,\ldots,t_n)\})$

where (i) $P(t_1,\ldots,t_n)$ is maximal in $C \lor P(t_1,\ldots,t_n) \lor P(t_1,\ldots,t_n)$ (ii) no literal is selected in $C \lor P(t_1,\ldots,t_n) \lor P(t_1,\ldots,t_n)$

Note that the superposition factoring rule differs from the resolution factoring rule in that it only applies to positive literals.

Definition 3.12.7 (Saturation)**.** A set N of clauses is called *saturated up to redundancy*, if any inference from non-redundant clauses in N yields a redundant clause with respect to N or is contained in N.

Examples for specific redundancy rules that can be efficiently decided are

Subsumption $(N \uplus \{C_1, C_2\}) \Rightarrow_{\text{SUP}} (N \cup \{C_1\})$
provided $C_1 \subset C_2$

Tautology Deletion $(N \uplus \{C \lor P(t_1,\ldots,t_n) \lor \neg P(t_1,\ldots,t_n)\}) \Rightarrow_{\text{SUP}} (N)$

Condensation $(N \uplus \{C_1 \lor L \lor L\}) \Rightarrow_{\text{SUP}} (N \cup \{C_1 \lor L\})$

Subsumption Resolution $(N \uplus \{C_1 \lor L, C_2 \lor \neg L\}) \Rightarrow_{\text{SUP}} (N \cup \{C_1 \lor L, C_2\})$
where $C_1 \subseteq C_2$

Proposition 3.12.8 (Completeness of the Reduction Rules)**.** All clauses removed by Subsumption, Tautology Deletion, Condensation and Subsumption Resolution are redundant with respect to the kept or added clauses.

Theorem 3.12.9 (Completeness)**.** Let N be a, possibly countably infinite, set of ground clauses. If N is saturated up to redundancy and $\bot \notin N$ then N is satisfiable and $N_{\mathcal{I}} \models N$.

Proof. The proof is by contradiction. So I assume: (i) for any clause D derived by Superposition Left or Factoring from N that D is redundant, i.e., $N^{\prec D} \models D$, (ii) $\bot \notin N$ and (iii) $N_{\mathcal{I}} \not\models N$. Then there is a minimal, with respect to $\prec$, clause $C \lor L \in N$ such that $N_{\mathcal{I}} \not\models C \lor L$ and L is a selected literal in $C \lor L$ or no literal in $C \lor L$ is selected and L is maximal. This clause must exist because $\bot \notin N$.

The clause $C \lor L$ is not redundant. For otherwise, $N^{\prec C \lor L} \models C \lor L$ and hence $N_{\mathcal{I}} \models C \lor L$, because $N_{\mathcal{I}} \models N^{\prec C \lor L}$, a contradiction.

I distinguish the case L is a positive and no literal selected in $C \lor L$ or L is a negative literal. Firstly, assume L is positive, i.e., $L = P(t_1,\ldots,t_n)$ for some ground atom $P(t_1,\ldots,t_n)$. Now if $P(t_1,\ldots,t_n)$ is strictly maximal in $C \lor P(t_1,\ldots,t_n)$ then actually $\delta_{C \lor P} = \{P(t_1,\ldots,t_n)\}$ and hence $N_{\mathcal{I}} \models C \lor P$, a contradiction. So $P(t_1,\ldots,t_n)$ is not strictly maximal. But then actually $C \lor P(t_1,\ldots,t_n)$ has the form $C_1' \lor P(t_1,\ldots,t_n) \lor P(t_1,\ldots,t_n)$ and Factoring derives $C_1' \lor P(t_1,\ldots,t_n)$ where $(C_1' \lor P(t_1,\ldots,t_n)) \prec (C_1' \lor P(t_1,\ldots,t_n) \lor P(t_1,\ldots,t_n))$. Now $C_1' \lor P(t_1,\ldots,t_n)$ is not redundant, strictly smaller than $C \lor L$, we have $C_1' \lor P(t_1,\ldots,t_n) \in N$ and $N_{\mathcal{I}} \not\models C_1' \lor P(t_1,\ldots,t_n)$, a contradiction against the choice that $C \lor L$ is minimal.

Secondly, let us assume L is negative, i.e., $L = \neg P(t_1,\ldots,t_n)$ for some ground atom $P(t_1,\ldots,t_n)$. Then, since $N_{\mathcal{I}} \not\models C \lor \neg P(t_1,\ldots,t_n)$ we know $P(t_1,\ldots,t_n) \in N_{\mathcal{I}}$. So there is a clause $D \lor P(t_1,\ldots,t_n) \in N$ where $\delta_{D \lor P(t_1,\ldots,t_n)} = \{P(t_1,\ldots,t_n)\}$ and $P(t_1,\ldots,t_n)$ is strictly maximal in $D \lor P(t_1,\ldots,t_n)$ and $(D \lor P(t_1,\ldots,t_n)) \prec (C \lor \neg P(t_1,\ldots,t_n))$. So Superposition Left derives $C \lor D$ where $(C \lor D) \prec (C \lor \neg P(t_1,\ldots,t_n))$. The derived clause

$C \vee D$ cannot be redundant, because for otherwise either $N^{\prec D \vee P(t_1,\ldots,t_n)} \models D \vee P(t_1,\ldots,t_n)$ or $N^{\prec C \vee \neg P(t_1,\ldots,t_n)} \models C \vee \neg P(t_1,\ldots,t_n)$. So $C \vee D \in N$ and $N_{\mathcal{I}} \not\models C \vee D$, a contradiction against the choice that $C \vee L$ is the minimal false clause. □

So the proof actually tells us that at any point in time we need only to consider either a superposition left inference between a minimal false clause and a productive clause or a factoring inference on a minimal false clause.

Theorem 3.12.10 (Compactness of First-Order Logic)**.** Let N be a, possibly countably infinite, set of first-order logic ground clauses. Then N is unsatisfiable iff there is a finite subset $N' \subseteq N$ such that N' is unsatisfiable.

Proof. If N is unsatisfiable, saturation via superposition generates $\bot$. So there is an i such that $N \Rightarrow^i_{\text{SUP}} N'$ and $\bot \in N'$. The clause $\bot$ is the result of at most i-many superposition inferences, reductions on clauses $\{C_1,\ldots,C_n\} \subseteq N$. Superposition is sound, so $\{C_1,\ldots,C_n\}$ is a finite, unsatisfiable subset of N. □

Corollary 3.12.11 (Compactness of First-Order Logic: Classical)**.** A set N of clauses is satisfiable iff all finite subsets of N are satisfiable.

Theorem 3.12.12 (Soundness and Completeness of Ground Superposition)**.** A first-order Σ-sentence ϕ is valid iff there exists a ground superposition refutation for $\operatorname{grd}(\Sigma, \operatorname{cnf}(\neg\phi))$.

Proof. A first-order sentence ϕ is valid iff $\neg\phi$ is unsatisfiable iff $\operatorname{acnf}(\neg\phi)$ is unsatisfiable iff $\operatorname{grd}(\Sigma, \operatorname{cnf}(\neg\phi))$ is unsatisfiable iff superposition provides a refutation of $\operatorname{grd}(\Sigma, \operatorname{cnf}(\neg\phi))$. □

Theorem 3.12.13 (Semi-Decidability of First-Order Logic by Ground Superposition)**.** If a first-order Σ-sentence ϕ is valid then a ground superposition refutation can be computed.

Proof. In a fair way enumerate $\operatorname{grd}(\Sigma, \operatorname{acnf}(\neg\phi))$ and perform superposition inference steps. The enumeration can, e.g., be done by considering Herbrand terms of increasing size. □

Example 3.12.14 (Ground Superposition)**.** Consider the below clauses 1-4 and superposition refutation with respect a KBO with precedence $P \succ Q \succ g \succ f \succ c \succ b \succ a$ where the weight function w returns 1 for all signature symbols. Maximal literals are marked with a *.

1.	$\neg P(f(c))^* \vee \neg P(f(c))^* \vee Q(b)$	(Input)
2.	$P(f(c))^* \vee Q(b)$	(Input)
3.	$\neg P(g(b,c))^* \vee \neg Q(b)$	(Input)
4.	$P(g(b,c))^*$	(Input)
5.	$\neg P(f(c))^* \vee Q(b)$	(Cond(1))
6.	$Q(b)^* \vee Q(b)^*$	(Sup(5,2)))
7.	$Q(b)^*$	(Fact(6))
8.	$\neg Q(b)^*$	(Sup(3,4))
10.	$\bot$	(Sup(8,7))

Note that clause 5 cannot be derived by Factoring whereas clause 7 can also be derived by Condensation. Clause 8 is also the result of a Subsumption Resolution application to clauses 3, 4.

Theorem 3.12.15 (Craig Theorem [24])**.** Let ϕ and ψ be two propositional (first-order ground) formulas so that $\phi \models \psi$. Then there exists a formula χ (called the *interpolant* for $\phi \models \psi$), so that χ contains only propositional variables (first-order signature symbols) occurring both in ϕ and in ψ so that $\phi \models \chi$ and $\chi \models \psi$.

Proof. Translate ϕ and $\neg\psi$ into CNF. Let N and M, respectively, denote the resulting clause set. Choose an atom ordering $\succ$ for which the propositional variables that occur in ϕ but not in ψ are maximal. Saturate N into N^* using $\Rightarrow_{\text{SUP}}$ with an empty selection function sel. Then saturate $N^* \cup M$ using $\Rightarrow_{\text{SUP}}$ to derive $\bot$. As N^* is already saturated, due to the ordering restrictions only inferences need to be considered where premises, if they are from N^*, only contain symbols that also occur in ψ. The conjunction of these premises is an interpolant χ. The theorem also holds for first-order formulas. For universal formulas the above proof can be easily extended. In the general case, a proof based on superposition technology is more complicated because of Skolemization. □