2.7 Propositional Superposition

Superposition was originally developed for first-order logic with equality [8]. Here I introduce its projection to propositional logic. Compared to the resolution calculus superposition adds (i) ordering and selection restrictions on inferences, (ii) an abstract redundancy notion, (iii) the notion of a partial model, based on the ordering for inference guidance, and (iv) a *saturation* concept.

Definition 2.7.1 (Clause Ordering)**.** Let $\prec$ be a total strict ordering on Σ. Then $\prec$ can be lifted to a total ordering on literals by $\prec \subseteq \prec_L$ and $P \prec_L \neg P$ and $\neg P \prec_L Q$, $\neg P \prec_L \neg Q$ for all $P \prec Q$. The ordering $\prec_L$ can be lifted to a total ordering on clauses $\prec_C$ by considering the multiset extension of $\prec_L$ for clauses.

For example, if $P \prec Q$, then $P \prec_L \neg P \prec_L Q \prec_L \neg Q$ and $P \vee Q \prec_C P \vee Q \vee Q \prec_C \neg Q$ because $\{P, Q\} \prec_L^{\text{mul}} \{P, Q, Q\} \prec_L^{\text{mul}} \{\neg Q\}$.

Proposition 2.7.2 (Properties of the Clause Ordering)**.** (i) The orderings on literals and clauses are total and well-founded.
(ii) Let C and D be clauses with $P = \text{atom}(\max(C))$, $Q = \text{atom}(\max(D))$, where $\max(C)$ denotes the maximal literal in C.

1. If $Q \prec_L P$ then $D \prec_C C$.
2. If $P = Q$, P occurs negatively in C but only positively in D, then $D \prec_C C$.

Eventually, I overload $\prec$ with $\prec_L$ and $\prec_C$. So if $\prec$ is applied to literals it denotes $\prec_L$, if it is applied to clauses, it denotes $\prec_C$. Note that $\prec$ is a total ordering on literals and clauses as well. Eventually we will restrict inferences to maximal literals with respect to $\prec$. For a clause set N, I define $N^{\prec C} = \{D \in N \mid D \prec C\}$.

Example 2.7.3 (Propositional Clause Ordering)**.** Let $P \prec Q \prec R \prec S$ and consider the clause set

$$N = \{P \vee \neg Q, Q \vee \neg R, P \vee \neg S, P \vee Q \vee S\}$$

then

$$\begin{array}{ll} N^{\prec C} = \emptyset & \text{if} \quad C = P \vee \neg Q \\ N^{\prec C} = \{P \vee \neg Q, Q \vee \neg R\} & \text{if} \quad C = S \\ N^{\prec C} = \{P \vee \neg Q, Q \vee \neg R, P \vee Q \vee S\} & \text{if} \quad C = \neg S \end{array}$$

Definition 2.7.4 (Abstract Redundancy)**.** A clause C is *redundant* with respect to a clause set N if $N^{\prec C} \models C$.

Tautologies are redundant. Subsumed clauses are redundant if $\subseteq$ is strict. Duplicate clauses are anyway eliminated quietly because the calculus operates on sets of clauses.

Note that for finite N, and any $C \in N$ redundancy $N^{\prec C} \models C$ can be decided but is as hard as testing unsatisfiability for a clause set N. So the goal is to invent redundancy notions that can be efficiently decided and that are useful.

Definition 2.7.5 (Selection Function)**.** The selection function sel maps clauses to one of its negative literals or $\bot$. If $\text{sel}(C) = \neg P$ then $\neg P$ is called *selected* in C. If $\text{sel}(C) = \bot$ then no literal in C is *selected*.

The selection function is, in addition to the ordering, a further means to restrict superposition inferences. If a negative literal is selected on a clause, any superposition inference must be on the selected literal.

Definition 2.7.6 (Partial Model Construction)**.** Given a clause set N and a total ordering $\prec$ we can construct a (partial) Herbrand model $N_{\mathcal{I}}$ for N inductively as follows:

$$\begin{array}{lcl} N_C & := & \bigcup_{D \prec C} \delta_D \\ \delta_D & := & \begin{cases} \{P\} & \text{if } D = D' \vee P, P \text{ strictly maximal, no literal} \\ & \text{selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases} \\ N_{\mathcal{I}} & := & \bigcup_{C \in N} \delta_C \end{array}$$

Clauses C with $\delta_C \neq \emptyset$ are called *productive*.

Proposition 2.7.7. Some properties of the partial model construction.

1. For every D with $(C \vee \neg P) \prec D$ we have $\delta_D \neq \{P\}$.
2. If $\delta_C = \{P\}$ then $N_C \cup \delta_C \models C$.
3. If $N_C \models D$ and $D \prec C$ then for all C' with $C \prec C'$ we have $N_{C'} \models D$ and in particular $N_{\mathcal{I}} \models D$.
4. There is no clause C with $P \vee P \prec C$ such that $\delta_C = \{P\}$.

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Please properly distinguish: N is a set of clauses interpreted as the conjunction of all clauses. $N^{\prec C}$ is of set of clauses from N strictly smaller than C with respect to $\prec$. $N_{\mathcal{I}}$, N_C are sets of atoms, often called *Herbrand Interpretations*. $N_{\mathcal{I}}$ is the overall (partial) model for N, whereas N_C is generated from all clauses from N strictly smaller than C. Validity is defined by $N_{\mathcal{I}} \models P$ if $P \in N_{\mathcal{I}}$ and $N_{\mathcal{I}} \models \neg P$ if $P \notin N_{\mathcal{I}}$, accordingly for N_C.

Given some clause set N, the partial model $N_{\mathcal{I}}$ can be extended to a valuation $\mathcal{A}$ by defining $\mathcal{A}(N_{\mathcal{I}}) := N_{\mathcal{I}} \cup \{\neg P \mid P \notin N_{\mathcal{I}}\}$. For some Herbrand interpretation $N_{\mathcal{I}}$ (N_C) I define $N_{\mathcal{I}} \models \phi$ if $\mathcal{A}(N_{\mathcal{I}})(\phi) = 1$.

Superposition Left $(N \uplus \{C_1 \vee P, C_2 \vee \neg P\}) \Rightarrow_{\text{SUP}} (N \cup \{C_1 \vee P, C_2 \vee \neg P\} \cup \{C_1 \vee C_2\})$

where (i) P is strictly maximal in $C_1 \vee P$ (ii) no literal in $C_1 \vee P$ is selected (iii) $\neg P$ is maximal and no literal selected in $C_2 \vee \neg P$, or $\neg P$ is selected in $C_2 \vee \neg P$

Factoring $(N \uplus \{C \lor P \lor P\}) \ \Rightarrow_{\text{SUP}} \ (N \cup \{C \lor P \lor P\} \cup \{C \lor P\})$
where (i) P is maximal in $C \lor P \lor P$ (ii) no literal is selected in $C \lor P \lor P$

Note that the superposition factoring rule differs from the resolution factoring rule in that it only applies to positive literals. Abstract redundancy can also be lifted to inferences, in the propositional case to Superposition Left applications. A Superposition Left inference

$$(N \uplus \{C_1 \lor P, C_2 \lor \neg P\}) \Rightarrow_{\text{SUP}} (N \cup \{C_1 \lor P, C_2 \lor \neg P\} \cup \{C_1 \lor C_2\})$$

is redundant if either one of the clauses $C_1 \lor P, C_2 \lor \neg P$ is redundant, or if $N^{\prec C_2 \lor \neg P} \models C_1 \lor C_2$. For a Factoring inference, the conclusion $C \lor P$ makes the premise $C \lor P \lor P$, so it is sufficient to require that $C \lor P \lor P$ is not redundant in order to guarantee $C \lor P$ to be non-redundant.

Definition 2.7.8 (Saturation)**.** A set N of clauses is called *saturated up to redundancy*, if any inference from non-redundant clauses in N yields a redundant clause with respect to N or is already contained in N.

Alternatively, saturation can be defined on the basis of redundant inferences. An superposition inference is called *redundant* if the inferred clause is redundant with respect to all clauses smaller than the maximal premise of the inference. Then a set N is saturated up to redundancy if all inferences from clauses from N are redundant.

Examples for specific redundancy rules that can be efficiently decided and are already well-known from the resolution calculus, Section 2.6, are

Subsumption $(N \uplus \{C_1, C_2\}) \ \Rightarrow_{\text{SUP}} \ (N \cup \{C_1\})$
provided $C_1 \subset C_2$

Tautology Deletion $(N \uplus \{C \lor P \lor \neg P\}) \ \Rightarrow_{\text{SUP}} \ (N)$

Condensation $(N \uplus \{C_1 \lor L \lor L\}) \ \Rightarrow_{\text{SUP}} \ (N \cup \{C_1 \lor L\})$

Subsumption Resolution $(N \uplus \{C_1 \lor L, C_2 \lor \text{comp}(L)\}) \ \Rightarrow_{\text{SUP}} \ (N \cup \{C_1 \lor L, C_2\})$
where $C_1 \subseteq C_2$

A clause C where Condensation is not applicable is called *condensed*.

Proposition 2.7.9. All clauses removed by Subsumption, Tautology Deletion, Condensation and Subsumption Resolution are redundant with respect to the kept or added clauses.

Corollary 2.7.10 (Soundness)**.** Superposition is sound.

Superposition is a refinement of resolution, so soundness is a consequence of the soundness part of Theorem 2.6.1.

Theorem 2.7.11 (Completeness)**.** If N is saturated up to redundancy and $\bot \notin N$ then N is satisfiable and $N_\mathcal{I} \models N$.

Proof. The proof is by contradiction. So I assume: (i) for any clause D derived by Superposition Left or Factoring from N that D is redundant, i.e., $N^{\prec D} \models D$, (ii) $\bot \notin N$ and (iii) $N_\mathcal{I} \not\models N$. Then there is a minimal, with respect to $\prec$, clause $C \vee L \in N$ such that $N_\mathcal{I} \not\models C \vee L$ and L is a selected literal in $C \vee L$ or no literal in $C \vee L$ is selected and L is maximal. This clause must exist because $\bot \notin N$.

The clause $C \vee L$ is not redundant. For otherwise, $N^{\prec C \vee L} \models C \vee L$ and hence $N_\mathcal{I} \models C \vee L$, because $N_\mathcal{I} \models N^{\prec C \vee L}$, a contradiction.

I distinguish the case L is a positive and no literal selected in $C \vee L$ or L is a negative literal. Firstly, assume L is positive, i.e., $L = P$ for some propositional variable P. Now if P is strictly maximal in $C \vee P$ then actually $\delta_{C \vee P} = \{P\}$ and hence $N_\mathcal{I} \models C \vee P$, a contradiction. So P is not strictly maximal. But then actually $C \vee P$ has the form $C'_1 \vee P \vee P$ and Factoring derives $C'_1 \vee P$ where $(C'_1 \vee P) \prec (C'_1 \vee P \vee P)$. Now $C'_1 \vee P$ is not redundant, strictly smaller than $C \vee L$, we have $C'_1 \vee P \in N$ and $N_\mathcal{I} \not\models C'_1 \vee P$, a contradiction against the choice that $C \vee L$ is minimal.

Secondly, let us assume L is negative, i.e., $L = \neg P$ for some propositional variable P. Then, since $N_\mathcal{I} \not\models C \vee \neg P$ we know $P \in N_\mathcal{I}$. So there is a clause $D \vee P \in N$ where $\delta_{D \vee P} = \{P\}$ and P is strictly maximal in $D \vee P$ and $(D \vee P) \prec (C \vee \neg P)$. So Superposition Left derives $C \vee D$ where $(C \vee D) \prec (C \vee \neg P)$. The derived clause $C \vee D$ cannot be redundant, because for otherwise either $N^{\prec D \vee P} \models D \vee P$ or $N^{\prec C \vee \neg P} \models C \vee \neg P$. So $C \vee D \in N$ and $N_\mathcal{I} \not\models C \vee D$, a contradiction against the choice that $C \vee L$ is the minimal false clause. □

So the proof actually tells us that at any point in time we need only to consider either a superposition left inference between a minimal false clause and a productive clause or a factoring inference on a minimal false clause.

The proof relies on the abstract redundancy notion and not on the specific redundancy rules introduced above. However, it also goes through on the basis of the concrete redundancy notions, see Exercise **??**.

According to Theorem 2.7.11 if a clause set N is saturated up to redundancy, the interpretation $N_\mathcal{I}$ is a model for N. This does not hold the other way round. If $N_\mathcal{I}$ is a model for N then N is not saturated, in general, see Exercise **??**.

I mentioned already that the abstract redundancy notion of superposition goes beyond the classical resolution reduction rules tautology deletion, subsumption, subsumption resolution and condensation. For example consider the clause set

$$N = \{\neg S \vee P,\ S \vee Q \vee \neg R,\ P \vee Q \vee \neg R\}$$

with ordering $S \prec P \prec Q \prec R$. Then $N^{\prec P \vee Q \vee \neg R} \models P \vee Q \vee \neg R$, i.e., the clause $P \vee Q \vee \neg R$ is redundant and can be deleted. This deletion is not justified by any of the classical resolution reduction rules.