# The Simplex Algorithm

The Simplex algorithm is a prominent algorithm for solving optimization problems over linear inequations. For automated reasoning, optimization is not the focus. Rather, solvability of a set of linear inequations as a subproblem of some theory combination is the typical application.

In this context the simplex algorithm is useful as well, due to its incremental nature. If an inequation $A$ is added to a set $N$ of inequations where the simplex algorithm has already found a solution for $N$, the algorithm needs not to start from scratch. Instead it continues with the solution found for $N$. In practice, we only need a few steps to derive a solution for $N \cup \{A\}$ if it exists.

# Preview CDCL(T)

For CDCL(T), we are only interested in the *existentially-quantified fragment* of a *theory T*. CDCL(T) extends a solver for the conjunctive fragment of a theory T (*theory solver*) to the *complete existentially-quantified fragment* of T. It even allows us to build a solver that combines different theories.

Important properties for a good theory solver for CDCL(T):

- can handle the conjunctive fragement of the theory
- good runtime
- produces an *assignment/model* in case of satisfiability
- good *incremental* behavior

# Drawbacks of Fourier-Motzkin

- worst case runtime $O(n^{2^m})$ (exponential runtime observed on relevant industrial problems)
- produces no *assignments* (would require additional bookkeeping)
- poor incremental behavior (would require additional expensive bookkeeping)

# The Simplex Algorithm

Idea: incrementally update a variable assignment until

a) the assignment is a solution, or
b) a conflict has been found

Advantages:

- worst case runtime single exponential (but very rare & not on relevant problems)
- provides an assignment or a conflict (with no overhead)
- good incremental behavior (just continue updating the assignment)

# The Input Problem

A set $N$ (conjunction) of (non-strict)[1] inequations over a set of variables $X$.

The inequations have the form:

$$(\sum_{x_j \in X} a_{i,j} x_j) \circ_i c_i,$$
$$\text{where } \circ_i \in \{\geq, \leq\} \text{ for all } i, \text{ and } \mathrm{gcd}\{a_{i,j} | x_j \in X\} = 1$$

Note that an equation $\sum a_i x_i = c$ can be encoded by two inequations $\{\sum a_i x_i \leq c, \sum a_i x_i \geq c\}$.

Additional assumptions (without loss of generality):

- we assume that the $x_j$ are all different
- we assume that the variables $x_j \in X$ are totally ordered by some ordering $\prec$[2]

[1] We will later describe how to handle strict inequalities.

[2] The ordering $\prec$ will eventually guarantee termination of the algorithm.

# The Goal

Decide whether there exists an assignment $\beta$ from the $x_j$ into $\mathbb{Q}$ such that $\text{LRA}(\beta) \models \bigwedge_i [(\sum_{x_j \in X} a_{i,j} x_j) \circ_i c_i]$, or equivalently, $\text{LRA}(\beta) \models N$

So the $x_j$ are free variables, i.e., placeholders for concrete values, i.e., existentially quantified.

# First Step: Transforming $N$

The first step is to transform $N$ into two disjoint sets $E$, $B$ of equations and simple bounds, respectively.

Hence, we split every inequation $\sum_{x_j \in X} a_{i,j} x_j \circ_i c_i$ from $N$ into:

- an equation $y_i \approx \sum_{x_j \in X} a_{i,j} x_j$ (moved to $E$), where $y_i$ is a fresh variable[3],
- a (simple) bound $y_i \circ_i c_i$ (moved to $B$)

Optimized Transformation:

- Just move simple bounds $x_i \circ_i c_i$ from $N$ to $B$.
- Use the same variable/equation for inequations with the same left hand side

[3] The $y_i$ are also part of the total ordering $\prec$ on all variables!

# Equivalence of the Transformation

Clearly, for any assignment $\beta$ and its respective extension on the $y_i$, the two representations are equivalent:

$$\mathrm{LRA}(\beta) \models N$$

iff

$$\mathrm{LRA}(\beta[y_i \mapsto \beta(\textstyle\sum_{x_j \in X} a_{i,j} x_j)]) \models E$$

and

$$\mathrm{LRA}(\beta[y_i \mapsto \beta(\textstyle\sum_{x_j \in X} a_{i,j} x_j)]) \models B.$$

# (In)dependent Variables

Given $E$ and $B$ a variable $z$ is called *dependent* if it occurs on the left hand side of an equation in $E$, i.e., there is an equation $(z \approx \sum_{x_j \in X} a_{i,j} x_j) \in E$. Otherwise, $z$ is called *independent*.

By construction the initial $y_i$ are all dependent and do not occur on the right hand side of an equation.

Note: when we write $(x \approx ay + t)$ for some equation, we always assume that $y \notin \mathrm{vars}(t)$.

# Pivot

Given:

- a dependant variable $x$,
- an independent variable $y$,
- a set of equations $E$, and
- the defining equation $(x \approx ay + t) \in E$ of $x$ with $a \neq 0$,

then the *pivot* operation exchanges the roles of $x$, $y$ in $E$, i.e., $x$ becomes independent and $y$ dependent.

Let $E'$ be $E$ without the defining equation of $x$. Then

$$\mathrm{piv}(E, x, y) := \left\{y \approx \frac{1}{a}x + \frac{1}{-a}t\right\} \cup E' \left\{y \mapsto \left(\frac{1}{a}x + \frac{1}{-a}t\right)\right\}.$$

# Update

Given:

- an assignment $\beta$,
- an independent variable $y$,
- a rational value $c$,
- a set of equations $E$

then the *update* of $\beta$ with respect to $y$, $c$, and $E$ is

$$\mathrm{upd}(\beta, y, c, E) := \beta[y \mapsto c, \{x \mapsto \beta[y \mapsto c](t) \mid x \approx t \in E\}].$$

# A Simplex State

A Simplex problem state is a quintuple $(E; B; \beta; S; s)$ where:

- $E$ is a set of equations,
- $B$ a set of simple bounds,
- $\beta$ an assignment to all variables in $E$, $B$,
- $S$ a set of derived bounds, and
- $s$ the status of the problem with $s \in \{\top, \mathrm{IV}, \mathrm{DV}, \bot\}$.

# The Status $s$

Given a state $(E; B; \beta; S; s)$:

- $s = \top$ indicates that $\text{LRA}(\beta) \models S$;
- $s = \text{IV}$ indicates that potentially $\text{LRA}(\beta) \not\models x \circ c$ for some independent variable $x$, $x \circ c \in S$;
- $s = \text{DV}$ indicates that $\text{LRA}(\beta) \models x \circ c$ for all independent variables $x$, $x \circ c \in S$, but potentially $\text{LRA}(\beta) \not\models x' \circ c'$ for some dependent variable $x'$, $x' \circ c' \in S$;
- $s = \bot$ indicates that the problem is unsatisfiable

# Start and Final States

| | |
|---|---|
| $(E; B; \beta_0; \emptyset; \top)$ | is the start state for $N$ and its transformation into $E$, $B$, and assignment $\beta_0(x) := 0$ for all $x \in \text{vars}(E \cup B)$ |
| $(E; \emptyset; \beta; S; \top)$ | is a final state, where $\text{LRA}(\beta) \models E \cup S$ and hence the problem is solvable |
| $(E; B; \beta; S; \bot)$ | is a final state, where $E \cup B \cup S$ has no model |

# Invariants

The important invariants of the simplex algorithm are:

i) for every dependent variable there is exactly one equation in $E$ defining the variable

ii) dependent variables do not occur on the right hand side of an equation

iii) $\mathrm{LRA}(\beta) \models E$.

These invariants hold initially and are maintained by a pivot ($\mathrm{piv}$) or an update ($\mathrm{upd}$) operation.

# Rough Draft

The simplex algorithm:

1. $\top$: moves one bound from $B$ to $S$
2. IV: fixes $\beta$ for all bounds in $S$ over independent variables (update)
3. DV: then fixes $\beta$ for all bounds in $S$ over dependent variables (pivot & update)
4. repeat

**FailBounds**

$$(E; B; \beta; S; \top) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \bot)$$

if there are two contradicting bounds $x \leq c_1$ and $x \geq c_2$ in $B \cup S$ for some variable $x$, i.e., $x \leq c_1$ and $x \geq c_2$ in $B \cup S$ with $c_1 < c_2$.

**Example:**
if $\{x \geq 5, x \leq 0\} \subseteq B \cup S$, then

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**EstablishBound**

$$(E; B \uplus \{x \circ c\}; \beta; S; \top) \;\Rightarrow_{\text{SIMP}}\; (E; B; \beta; S \cup \{x \circ c\}; \text{IV})$$

**Example:**

$$E := \left\{ \begin{array}{l} u \approx x + 2y, \\ v \approx x - y \end{array} \right\}, \quad \begin{array}{ll} B & := \{x \geq 0, y \leq -1, u \geq 1, v \geq 2, v \leq 3\} \\ \beta & := \{x \mapsto 0, y \mapsto 0, u \mapsto 0, v \mapsto 0\} \\ S & := \{\} \end{array}$$

$$(E; B; \beta; \{\}; \top) \quad \Rightarrow_{\text{SIMP}} \quad (E; B \setminus \{x \geq 0\}; \beta; \{x \geq 0\}; \text{IV})$$

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**AckBounds**

$$(E; B; \beta; S; V) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \top)$$

if $\text{LRA}(\beta) \models S$, $V \in \{\text{IV}, \text{DV}\}$

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## FixIndepVar

$$(E; B; \beta; S; \text{IV}) \Rightarrow_{\text{SIMP}} (E; B; \text{upd}(\beta, x, c, E); S; \text{IV})$$

if $(x \circ c) \in S$, $\text{LRA}(\beta) \not\models x \circ c$, $x$ independent

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$$(E; B; \beta; S; \mathrm{IV}) \;\Rightarrow_{\mathrm{SIMP}}\; (E; B; \beta; S; \mathrm{DV})$$

if $\mathrm{LRA}(\beta) \models x \circ c$, for all independent variables $x$ with bounds $x \circ c$ in $S$

**Example:**

$$E := \left\{ \begin{array}{l} u \approx x + 2y, \\ v \approx x - y \end{array} \right\}, \begin{array}{ll} B & := \{v \geq 2, v \leq 3\} \\ \beta & := \{x \mapsto 0, y \mapsto -1, u \mapsto -2, v \mapsto 1\} \\ S & := \{x \geq 0, y \leq -1, u \geq 1\} \end{array}$$

$$(E; B; \beta; S; \mathrm{IV}) \quad \Rightarrow^{\mathrm{AckIB.}}_{\mathrm{SIMP}} \quad (E; B; \beta; S; \mathrm{DV})$$

**AckIndepBound**

$$(E; B; \beta; S; \mathrm{IV}) \Rightarrow_{\mathrm{SIMP}} (E; B; \beta; S; \mathrm{DV})$$

if $\mathrm{LRA}(\beta) \models x \circ c$, for all independent variables $x$ with bounds $x \circ c$ in $S$

**Example:**

$$E := \left\{ \begin{array}{l} u \approx x + 2y, \\ v \approx x - y \end{array} \right\}, \quad \begin{array}{ll} B & := \{v \geq 2, v \leq 3\} \\ \beta & := \{x \mapsto 0, y \mapsto -1, u \mapsto -2, v \mapsto 1\} \\ S & := \{x \geq 0, y \leq -1, u \geq 1\} \end{array}$$

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$$(E; B; \beta; S; \mathrm{IV}) \Rightarrow^{\mathrm{AckIB.}}_{\mathrm{SIMP}} (E; B; \beta; S; \mathrm{DV})$$

**FixDepVar**$\geq$

$$(E; B; \beta; S; \mathrm{DV}) \Rightarrow_{\mathrm{SIMP}} (E'; B; \mathtt{upd}(\beta, x, c, E'); S; \mathrm{DV})$$

if $(x \geq c) \in S$, $x$ dependent, $\mathrm{LRA}(\beta) \not\models x \geq c$, there is an independent variable $y$ and equation $(x \approx ay + t) \in E$ where $(a > 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a < 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$ and $E' := \mathrm{piv}(E, x, y)$

**Example:**

$$E := \left\{ \begin{array}{l} u \approx x + 2y, \\ v \approx x - y \end{array} \right\}, \begin{array}{ll} B & := \{v \geq 2, v \leq 3\} \\ \beta & := \{x \mapsto 0, y \mapsto -1, u \mapsto -2, v \mapsto 1\} \\ S & := \{x \geq 0, y \leq -1, u \geq 1\} \end{array}$$

$$E' := \left\{ \begin{array}{l} x \approx u - 2y, \\ v \approx u - 3y \end{array} \right\}, \begin{array}{ll} \beta' & := \mathtt{upd}(\beta, u, 1, E') \\ & := \{u \mapsto 1, y \mapsto -1, x \mapsto 3, v \mapsto 4\} \end{array}$$

**FixDepVar$\geq$**

$$(E; B; \beta; S; \text{DV}) \;\Rightarrow_{\text{SIMP}}\; (E'; B; \text{upd}(\beta, x, c, E'); S; \text{DV})$$

if $(x \geq c) \in S$, $x$ dependent, $\text{LRA}(\beta) \not\models x \geq c$, there is an independent variable $y$ and equation $(x \approx ay + t) \in E$ where
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**FixDepVar$\geq$**

$$(E; B; \beta; S; \text{DV}) \Rightarrow_{\text{SIMP}} (E'; B; \text{upd}(\beta, x, c, E'); S; \text{DV})$$

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**FixDepVar≥**

$$(E; B; \beta; S; \mathrm{DV}) \;\Rightarrow_{\mathrm{SIMP}}\; (E'; B; \mathrm{upd}(\beta, x, c, E'); S; \mathrm{DV})$$

if $(x \geq c) \in S$, $x$ dependent, $\mathrm{LRA}(\beta) \not\models x \geq c$, there is an independent variable $y$ and equation $(x \approx ay + t) \in E$ where
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$$(E; B; \beta; S; \mathrm{DV}) \Rightarrow_{\mathrm{SIMP}} (E'; B; \mathrm{upd}(\beta, x, c, E'); S; \mathrm{DV})$$

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**FixDepVar$\geq$**

$$(E; B; \beta; S; \mathrm{DV}) \;\Rightarrow_{\mathrm{SIMP}}\; (E'; B; \mathrm{upd}(\beta, x, c, E'); S; \mathrm{DV})$$

if $(x \geq c) \in S$, $x$ dependent, $\mathrm{LRA}(\beta) \not\models x \geq c$, there is an independent variable $y$ and equation $(x \approx ay + t) \in E$ where $(a > 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a < 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$ and $E' := \mathrm{piv}(E, x, y)$

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**FixDepVar**$\leq$

$$(E; B; \beta; S; \text{DV}) \;\Rightarrow_{\text{SIMP}}\; (E'; B; \mathtt{upd}(\beta, x, c, E'); S; \text{DV})$$

if $(x \leq c) \in S$, $x$ dependent, $\text{LRA}(\beta) \not\models x \leq c$, there is an independent variable $y$ and equation $(x \approx ay + t) \in E$ where $(a < 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a > 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$ and $E' := \text{piv}(E, x, y)$

**FailDepVar$\leq$**

$$(E; B; \beta; S; \mathrm{DV}) \ \Rightarrow_{\mathrm{SIMP}} \ (E; B; \beta; S; \bot)$$

if $(x \leq c) \in S$, $x$ dependent, $\mathrm{LRA}(\beta) \not\models x \leq c$ and there is no independent variable $y$ and equation $(x \approx ay + t) \in E$ where $(a < 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a > 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$

**Example:**

$$E := \left\{ \begin{array}{l} x \approx u - 2y, \\ v \approx u - 3y \end{array} \right\}, \quad \begin{array}{ll} B & := \{v \geq 2\} \\ \beta & := \{u \mapsto 1, y \mapsto -1, x \mapsto 3, v \mapsto 4\} \\ S & := \{x \geq 0, y \leq -1, u \geq 1, v \leq 3\} \end{array}$$

**FailDepVar≤**

$$(E; B; \beta; S; \mathrm{DV}) \;\Rightarrow_{\mathrm{SIMP}}\; (E; B; \beta; S; \bot)$$

if $(x \leq c) \in S$, $x$ dependent, $\mathrm{LRA}(\beta) \not\models x \leq c$ and there is no independent variable $y$ and equation $(x \approx ay + t) \in E$ where ($a < 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S$) or ($a > 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S$)

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**FailDepVar$\leq$**

$$(E; B; \beta; S; \mathrm{DV}) \;\Rightarrow_{\mathrm{SIMP}}\; (E; B; \beta; S; \bot)$$

if $(x \leq c) \in S$, $x$ dependent, $\mathrm{LRA}(\beta) \not\models x \leq c$ and there is no independent variable $y$ and equation $(x \approx ay + t) \in E$ where
$(a < 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or
$(a > 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$

**Example:**

$$E := \left\{ \begin{array}{l} x \approx u - 2y, \\ v \approx u - 3y \end{array} \right\}, \quad \begin{array}{ll} B & := \{v \geq 2\} \\ \beta & := \{u \mapsto 1, y \mapsto -1, x \mapsto 3, v \mapsto 4\} \\ S & := \{x \geq 0, y \leq -1, u \geq 1, v \leq 3\} \end{array}$$

**FailDepVar$\geq$**

$$(E; B; \beta; S; \mathrm{DV}) \;\Rightarrow_{\mathrm{SIMP}}\; (E; B; \beta; S; \bot)$$

if $(x \geq c) \in S$, $x$ dependent, $\beta \not\models_{\mathrm{LA}} x \geq c$ and there is no independent variable $y$ and equation $(x \approx ay + t) \in E$ where (if $a > 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S$) or (if $a < 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S$)

**List of rules:**

**EstablishBound** $(E; B \uplus \{x \circ c\}; \beta; S; \top) \Rightarrow_{\text{SIMP}} (E; B; \beta; S \cup \{x \circ c\}; \text{IV})$

**AckBounds** $(E; B; \beta; S; s) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \top)$
if $\text{LRA}(\beta) \models S$, $s \in \{\text{IV}, \text{DV}\}$

**FixIndepVar** $(E; B; \beta; S; \text{IV}) \Rightarrow_{\text{SIMP}} (E; B; \text{upd}(\beta, x, c, E); S; \text{IV})$
if $(x \circ c) \in S$, $\text{LRA}(\beta) \not\models x \circ c$, $x$ independent

**AckIndepBound** $(E; B; \beta; S; \mathrm{IV}) \Rightarrow_{\mathrm{SIMP}} (E; B; \beta; S; \mathrm{DV})$

if $\mathrm{LRA}(\beta) \models x \circ c$, for all independent variables $x$ with bounds $x \circ c$ in $S$

**FixDepVar**$\leq$$(E; B; \beta; S; \mathrm{DV}) \Rightarrow_{\mathrm{SIMP}} (E'; B; \mathrm{upd}(\beta, x, c, E'); S; \mathrm{DV})$

if $(x \leq c) \in S$, $x$ dependent, $\mathrm{LRA}(\beta) \not\models x \leq c$, there is an independent variable $y$ and equation $(x \approx ay + t) \in E$ where $(a < 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a > 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$ and $E' := \mathrm{piv}(E, x, y)$

**FixDepVar**$\geq$$(E; B; \beta; S; \mathrm{DV}) \Rightarrow_{\mathrm{SIMP}} (E'; B; \mathrm{upd}(\beta, x, c, E'); S; \mathrm{DV})$

if $(x \geq c) \in S$, $x$ dependent, $\mathrm{LRA}(\beta) \not\models x \geq c$, there is an independent variable $y$ and equation $(x \approx ay + t) \in E$ where $(a > 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a < 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$ and $E' := \mathrm{piv}(E, x, y)$

**FailBounds** $(E; B; \beta; S; \top) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \bot)$

if there are two contradicting bounds $x \le c_1$ and $x \ge c_2$ in $B \cup S$ for some variable $x$

**FailDepVar$\le$** $(E; B; \beta; S; \text{DV}) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \bot)$

if $(x \le c) \in S$, $x$ dependent, $\text{LRA}(\beta) \not\models x \le c$ and there is no independent variable $y$ and equation $(x \approx ay + t) \in E$ where $(a < 0$ and $\beta(y) < c'$ for all $(y \le c') \in S)$ or $(a > 0$ and $\beta(y) > c'$ for all $(y \ge c') \in S)$

**FailDepVar$\ge$** $(E; B; \beta; S; \text{DV}) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \bot)$

if $(x \ge c) \in S$, $x$ dependent, $\text{LRA}(\beta) \not\models x \ge c$ and there is no independent variable $y$ and equation $(x \approx ay + t) \in E$ where (if $a > 0$ and $\beta(y) < c'$ for all $(y \le c') \in S)$ or (if $a < 0$ and $\beta(y) > c'$ for all $(y \ge c') \in S)$

## 6.2.7 Lemma (Simplex State Invariants)

The following invariants hold for any state $(E_i; B_i; \beta_i; S_i; s_i)$ derived by $\Rightarrow_{\text{SIMP}}$ on a start state $(E_0; B_0; \beta_0; \emptyset; \top)$:

(i) for every dependent variable there is exactly one equation in $E$ defining the variable

(ii) dependent variables do not occur on the right hand side of an equation

(iii) $\text{LRA}(\beta) \models E_i$

(iv) for all independant variables $x$ either $\beta_i(x) = 0$ or $\beta_i(x) = c$ for some bound $x \circ c \in S_i$

(v) for all assignments $\alpha$ it holds $\text{LRA}(\alpha) \models E_0$ iff $\text{LRA}(\alpha) \models E_i$

## 6.2.8 Lemma (Simplex Run Invariants)

For any run of $\Rightarrow_{\text{SIMP}}$ from start state
$(E_0; B_0; \beta_0; \emptyset; \top) \Rightarrow_{\text{SIMP}} (E_1; B_1; \beta_1; S_1; s_1) \Rightarrow_{\text{SIMP}} \ldots$:

(i) the set $\{\beta_o, \beta_1, \ldots\}$ is finite

(ii) if the sets of dependent and independent variables for two equational systems $E_i$, $E_j$ coincide, then $E_i = E_j$

(iii) the set $\{E_o, E_1, \ldots\}$ is finite

(iv) let $S_i$ not contain contradictory bounds, then $(E_i; B_i; \beta_i; S_i; s_i) \Rightarrow_{\text{SIMP}}^{\text{FIV},*}$ is finite

## 6.2.9 Corollary (Infinite Runs Contain a Cycle)

Let $(E_0; B_0; \beta_0; \emptyset; \top) \Rightarrow_{\text{SIMP}} (E_1; B_1; \beta_1; S_1; s_1) \Rightarrow_{\text{SIMP}} \ldots$ be an infinite run. Then there are two states $(E_i; B_i; \beta_i; S_i; s_i)$, $(E_k; B_k; \beta_k; S_k; s_k)$ such that $i \neq k$ and $(E_i; B_i; \beta_i; S_i; s_i) = (E_k; B_k; \beta_k; S_k; s_k)$.

### 6.2.10 Definition (Reasonable Strategy)

A *reasonable* strategy prefers FailBounds over EstablishBounds and the FixDepVar rules select minimal variables $x$, $y$ in the ordering $\prec$.

## 6.2.11 Theorem (Simplex Soundness, Completeness & Termination)

Given a reasonable strategy and initial set $N$ of inequations and its separation into $E$ and $B$ :

(i) $\Rightarrow_{\text{SIMP}}$ terminates on $(E; B; \beta_0; \emptyset; \top)$,

(ii) if $(E; B; \beta_0; \emptyset; \top) \Rightarrow^*_{\text{SIMP}} (E'; B'; \beta; S; \bot)$ then $N$ has no solution,

(iii) if $(E; B; \beta_0; \emptyset; \top) \Rightarrow^*_{\text{SIMP}} (E'; \emptyset; \beta; B; \top)$ and $(E; \emptyset; \beta; B; \top)$ is a normal form, then $\text{LRA}(\beta) \models N$,

(iv) all final states $(E'; B'; \beta; S; s)$ match either (ii) or (iii).

In case of strict bounds the idea is to introduce an infinitesimal small constant $\delta > 0$ and to replace the strict bound by a non-strict one. So, for example, a bound $x < 5$ is replaced by $x \leq 5 - \delta$. Now $\delta$ is treated symbolically through the overall computation, i.e., we extend $\mathbb{Q}$ to $\mathbb{Q}_\delta$ with new pairs $(q, k)$ with $q, k \in \mathbb{Q}$ where $(q, k)$ represents $q + k\delta$ and the operations, relations on $\mathbb{Q}$ are lifted to $\mathbb{Q}_\delta$:

$$
\begin{aligned}
(q_1, k_1) + (q_2, k_2) &:= (q_1 + q_2, k_1 + k_2) \\
p(q, k) &:= (pq, pk) \\
(q_1, k_1) \leq (q_2, k_2) &:= (q_1 < q_2) \vee (q_1 = q_2 \wedge k_1 \leq k_2)
\end{aligned}
$$