# Automated Reasoning

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# CDCL Extensions 2.15

- Cost optimal models OCDCL

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- MaxSAT
- minimal covering models
- chronological CDCL

# Computing Cost Optimal Models (OCDCL)

$N = \{A \vee B,\ A \vee \neg B\}$

$cost : \{A, \neg A, B, \neg B\} \to \mathbb{R}^+$

$cost(A) = cost(B) = 1$

$cost(\neg A) = cost(\neg B) = 0$

$M = \{AB,\ A\neg B\}$

$cost(AB) = 2$

$\boxed{cost(A \neg B) = 1}$

# OCDCL States

| | |
|---|---|
| $(\epsilon; N; \emptyset; 0; \top; \epsilon)$ | start state for some clause set $N$ |
| $(M; N; U; k; \bot; O)$ | final state, where<br>■ $N$ has no model if $O = \epsilon$<br>■ $O$ is a cost optimal model if $O \neq \epsilon$ |
| $(M; N; U; k; \top; O)$ | intermediate model search state |
| $(M; N; U; k; D; O)$ | backtracking state if $D \notin \{\top, \bot\}$ |

- $O$ denotes the cost optimal model of $N$
- $M$, $N$, $U$, $k$, $D$ are defined analogously to CDCL
- but OCDCL always terminates with $D = \bot$

# OCDCL Rules

**Propagate** $(M; N; U; k; \top; O) \Rightarrow_{\text{OCDCL}} (ML^{C\vee L}; N; U; k; \top; O)$
provided $C \vee L \in (N \cup U)$, $M \models \neg C$, $L$ is undefined in $M$

**Decide** $(M; N; U; k; \top; O) \Rightarrow_{\text{OCDCL}}$
$(ML^{k+1}; N; U; k+1; \top; O)$
provided $L$ is undefined in $M$, contained in $N$

**ConflSat** $(M; N; U; k; \top; O) \Rightarrow_{\text{OCDCL}} (M; N; U; k; D; O)$
provided $D \in (N \cup U)$ and $M \models \neg D$

**ConflOpt** $(M; N; U; k; \top; O) \Rightarrow_{\text{OCDCL}} (M; N; U; k; \neg M; O)$
provided $O \neq \epsilon$ and $\mathrm{cost}(M) \geq \mathrm{cost}(O)$

# OCDCL Rules (ctd.)

**Skip** $(ML^{C\vee L}; N; U; k; D; O) \Rightarrow_{\text{OCDCL}} (M; N; U; k; D; O)$
provided $D \notin \{\top, \bot\}$ and $\mathsf{comp}(L)$ does not occur in $D$

**Resolve** $(ML^{C\vee L}; N; U; k; D \vee \mathsf{comp}(L); O) \Rightarrow_{\text{OCDCL}}$
$(M; N; U; k; D \vee C; O)$
provided $D$ is of level $k$

**Backtrack** $(M_1 K^{i+1} M_2; N; U; k; D \vee L; O) \Rightarrow_{\text{OCDCL}}$
$(M_1 L^{D\vee L}; N; U \cup \{D \vee L\}; i; \top; O)$
provided $L$ is of level $k$ and $D$ is of level $i$

**Improve** $(M; N; U; k; \top; O) \Rightarrow_{\text{OCDCL}} (M; N; U; k; \top; M)$
provided $M \models N$, $M$ is total, i.e., contains all atoms in $N$, and $O = \epsilon$ or $\mathsf{cost}(M) < \mathsf{cost}(O)$

### 2.15.1 Definition (Reasonable OCDCL Strategy)

An OCDCL strategy is *reasonable* if ConflSat is preferred over ConflOpt is preferred over Improve is preferred over Propagate which is preferred over the remaining rules.

## 2.15.3 Proposition (OCDCL Basic Properties)

Consider an OCDCL state $(M; N; U; k; D'; O)$ derived by a reasonable strategy from start state $(\epsilon, N, \emptyset, 0, \top, \epsilon)$. Then the following properties hold:

1. $M$ is consistent.
2. If $O \neq \epsilon$ then $O$ is consistent and $O \models N$.
3. If $D' \notin \{\top, \bot\}$ then $M \models \neg D'$.
4. If $D' \notin \{\top, \bot\}$ then (i) $D'$ is entailed by $N \cup U$, or (ii) for any model $M' \models \{\neg D'\} \cup N \cup U$: $\mathrm{cost}(M') \geq \mathrm{cost}(O)$.
5. If $D' = \top$ and $M$ contains only propagated literals then for each valuation $\mathcal{A}$ with $\mathcal{A} \models (N \cup U)$ it holds $\mathcal{A} \models M$.

## 2.15.3 Proposition (OCDCL Basic Properties (ctd.))

6. For all models $M$ with $M \models N$: if $O = \epsilon$ or $\mathrm{cost}(M) < \mathrm{cost}(O)$ then $M \models (N \cup U)$.
7. If $D' = \bot$ then OCDCL terminates and there is no model $M'$ with $M' \models N$ and $\mathrm{cost}(M') < \mathrm{cost}(O)$.
8. Each infinite derivation

$$(\epsilon; N; \emptyset; 0; \top; \epsilon) \Rightarrow_{\text{OCDCL}} (M_1; N; U_1; k_1; D_1; O_1) \Rightarrow_{\text{OCDCL}} \cdots$$

   contains an infinite number of Backtrack applications.
9. OCDCL never learns the same clause twice.

## 2.15.4 Lemma (OCDCL Normal Forms)

The OCDCL calculus with a reasonable strategy has only 2 normal forms:

- $(M; N; U; 0; \bot; O)$ where $O \neq \epsilon$, $O \models N$ and $\mathrm{cost}(O)$ is optimal
- $(M; N; U; 0; \bot; \epsilon)$ where $N$ is unsatisfiable

## 2.15.5 Lemma (OCDCL Termination)

OCDCL with a reasonable strategy terminates in a state $(M; N; U; 0; \bot; O)$.

## 2.15.6 Theorem (OCDCL Correctness)

OCDCL with a reasonable strategy starting from a state $(\epsilon; N; \emptyset; 0; \top; \epsilon)$ terminates in a state $(M; N; U; 0; \bot; O)$. If $O = \epsilon$ then $N$ is unsatisfiable. If $O \neq \epsilon$ then $O \models N$ and for any other model $M'$ with $M' \models N$ it holds $\mathsf{cost}(M') \geq \mathsf{cost}(O)$.

# Improving OCDCL

**Prune** $(M; N; U; k; \top; O) \Rightarrow_{\text{OCDCL}} (M; N; U; k; \neg M; O)$
provided for all total trail extensions $MM'$ of $M$ it holds $\text{cost}(MM') \geq \text{cost}(O)$

**ConflOpt** $(M; N; U; k; \top; O) \Rightarrow_{\text{OCDCL}} (M; N; U; k; \neg M; O)$
provided $O \neq \epsilon$ and $\text{cost}(M) \geq \text{cost}(O)$

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provided $O \neq \epsilon$ and $\text{cost}(M) \geq \text{cost}(O)$

# The Max-SAT Problem

Given $N = N_H \uplus N_S$ where $N_H$ are hard clauses and $N_S$ are soft clauses

Find $\mathcal{A} \models N_H$ with minimal cost $\sum_{\mathcal{A}\models\neg C}^{C\in N_S} \omega(C)$ where $\omega\colon N_S \mapsto \mathbb{R}^+$

$N_H = \{A \vee B\}$ $N_S = \{\neg A \vee C, B \vee C\}$

$\omega(\neg A \vee C) = 1$ $\omega(ABC) = 0$

$\omega(B \vee C) = 2$ $\omega(A \neg B \neg C) = 3$

# Reducing Max-SAT to OCDCL

1. Introduce a fresh variable $S_i$ for each $C_i \in N_S = \{C_1, \ldots, C_n\}$
2. Define $N'_S = \{S_i \vee C_i \mid C_i \in N_S\}$
3. Compute cost optimal model for $N' = N_H \uplus N'_S$ with cost function $\mathrm{cost}(L) = \begin{cases} \omega(C_i) & \text{if } L = S_i \\ 0 & \text{otherwise} \end{cases}$

$N_H = \{A \vee B\}$ $\quad N_S = \{\underbrace{\neg A \vee C}_{C_1}, \underbrace{B \vee C}_{C_2}\}$ $\quad \mathrm{cost}(S_1) = 1$

$\mathrm{cost}(S_2) = 2$

$N'_S = \{S_1 \vee \neg A \vee C,$

$S_2 \vee B \vee C\}$ $\quad N' = \{A \vee B, S_1 \vee \neg A \vee C, S_2 \vee B \vee C\}$

$A \neg B \neg C\ S_1 S_2$

## 2.15.7 Theorem (Max-SAT Solution)

$\mathcal{A}$ is a Max-SAT solution for $N = N_H \uplus N_S$ with minimal value $c = \sum_{\mathcal{A} \models \neg C}^{C \in N_S} \omega(C)$ iff $(\epsilon; N'; \emptyset; 0; \top; \epsilon) \Rightarrow^{*}_{\text{OCDCL}} (M; N'; U; k; \bot; O)$ with a reasonable strategy where $N' = N_H \uplus N'_S$, and $\text{cost}(O) = c$.

# Optimization

1. Introduce a fresh variable $S_i$ for each $C_i \in N_S = \{C_1, \ldots, C_n\}$
2. Define $N'_S = \{S_i \vee C_i \mid C_i \in N_S\} \cup \{\neg C_i \vee \neg S_i \mid C_i \in N_S\}$
3. Compute cost optimal model for $N' = N_H \uplus N'_S$ with cost function $\mathrm{cost}(L) = \begin{cases} \omega(C_i) & \text{if } L = S_i \\ 0 & \text{otherwise} \end{cases}$

# Minimal Covering Models

Given $\mathcal{M}$ set of all models of the set of clauses $N$

Find $\mathcal{M}' \subseteq \mathcal{M}$ such that

- $|\mathcal{M}'|$ is minimal
- for each propositional variable $P$ in $N$ there is a model $M \in \mathcal{M}'$ with $M(P) = 1$

$N = \{A \vee B, \neg A \vee C, B \vee C\}$

$\mathcal{M} = \{ABC, A\neg BC, \neg ABC, \neg AB\neg C\}$

$\mathcal{M}' = \{\underline{A}\underline{B}\underline{C}\}$ $\mathcal{M}'' = \{\underline{A}\neg B\underline{C}, \neg A\underline{B}C\}$

# Reduction to OCDCL

Given $N$ with variables $P_1, \ldots P_n$ and clauses $C_1, \ldots, C_m$

1. Define $N_j := \{C\{P_i \mapsto P_i^j \mid 1 \le i \le n\} \vee \neg Q_j \mid C \in N\}$
2. Define $N_+ := \{P_i^1 \vee \ldots \vee P_i^n \mid 1 \le i \le n\}$
3. Define $N_Q := \{\neg P_i^j \vee Q_j \mid 1 \le i, j \le n\}$
4. Find a minimal cost model of $(\cup_{j=1}^n N_j) \cup N_+ \cup N_Q$ with cost function $\mathrm{cost}(M) = \sum_{j=1}^n M(Q_j)$

Requires

- $O(n^2)$ additional variables
- $O(n \cdot \max(m, n))$ additional clauses

Note: $n$ = upper bound of number of models (Algorithm 10)

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Requires

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Note: $n$ = upper bound of number of models (Algorithm 10)

# Chronological CDCL

Motivation: Reduce repeating assignments after backtracking

Main Idea: Backtrack chronologically after conflict analysis

A. Nadel and V. Ryvchin, “Chronological Backtracking”, SAT’18.
S. Möhle and A. Biere, “Backing Backtracking”, SAT’19.

# CDCL Invariants

1. The assignment trail contains neither complementary pairs of literals nor duplicates.
2. The assignment trail preceding the current decision level does not falsify the formula.
3. On every decision level preceding the current decision level all unit clauses are propagated until completion.
4. The literals are ordered on the assignment trail in ascending order with respect to their decision level.
5. At decision levels greater than zero the conflicting clause contains at least two literals with the current decision level.

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5. At decision levels greater than zero the conflicting clause contains at least two literals with the current decision level.

$\longrightarrow$ violated by Chronological CDCL $\ddot{\frown}$

# Backtracking Chronologically

| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $M$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

✗

conflicting { -47 -17 -44 }

learned { -30 -47 -18 23 }

# Backtracking Chronologically

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $M$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

✗

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| conflicting | { | | | | -47 | | | | | | | | | | -17 | -44 | } |
| learned | { | | | -30 | -47 | | -18 | | | | | | | 23 | | | } |

# Backtracking Chronologically

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $M$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |
| conflicting | | { | | | -47 | | | | | | | | | | -17 | -44 | } |
| learned | | { | | -30 | -47 | | -18 | | | | | | | 23 | | | } |

✗

# Backtracking Chronologically

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | |
| $M$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 | ✗ |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | |

conflicting { -47 -17 -44 }

learned { -30 -47 -18 23 }

## Out-of-Order Propagation

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $M$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| $M$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 23 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 4 |

# Out-of-Order Propagation

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $M$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| $M$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 23 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 4 |
| | | | | | | | | | | | | $\uparrow$ |

## One Single Literal at Conflict Level

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | |
| $M$ | $\cdots$ | 18 | 6 | -7 | -8 | 45 | 23 | -38 | -9 | 10 | -11 | 13 | 16 | -17 | -25 | 42 | 12 | -41 | ✗ |
| $\delta$ | $\cdots$ | 4 | 5 | 5 | 5 | 5 | 4 | 4 | 5 | 6 | 7 | 5 | 4 | 4 | 4 | 4 | 5 | 5 | |

conflict { 17, -42, -12 }

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| $M$ | $\cdots$ | 18 | 23 | -38 | 16 | -17 | -25 | 42 | -12 |
| $\delta$ | $\cdots$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |

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| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $M$ | $\cdots$ | 18 | 6 | -7 | -8 | 45 | 23 | -38 | -9 | 10 | -11 | 13 | 16 | -17 | -25 | 42 | 12 | -41 |
| $\delta$ | $\cdots$ | 4 | 5 | 5 | 5 | 5 | 4 | 4 | 5 | 6 | 7 | 5 | 4 | 4 | 4 | 4 | 5 | 5 |

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conflict { 17, -42, -12 }

| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|
| $M$ | $\cdots$ | 18 | 23 | -38 | 16 | -17 | -25 | 42 | -12 |
| $\delta$ | $\cdots$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |

# CDCLChrono States

| | |
|---|---|
| $(\epsilon; N; \emptyset; \delta_\infty; D)$ | start state for some clause set $N$ |
| $(M; N; U; \delta; \top)$ | final state if $M \models N$ and all literals from $N$ are defined in $M$ |
| $(M; N; U; \delta; \bot)$ | final state where $N$ has no model |
| $(M; N; U; k; \delta; D)$ | intermediate model search state if $M \not\models N$ |
| $(M; N; U; k; \delta, D)$ | backtracking state if $D \notin \{\top, \bot\}$ |

- $\delta: \mathrm{fvars}(N) \mapsto \mathbb{N} \cup \{\infty\}$ denotes the decision level function
- $\delta_\infty$ denotes the decision level function where all literals are unassigned, i.e., assigned decision level $\infty$

# CDCLChrono Rules

**Propagate** $(M; N; U; \delta; \top) \Rightarrow_{\text{CDCLChrono}}$ $(ML^{C\vee L}; N; U; \delta[L \mapsto k]; \top)$

provided $C \vee L \in (N \cup U)$, $M \models \neg C$, $L$ is undefined in $M$, and $C$ is of level $k$

**Decide** $(M; N; U; \delta; \top) \Rightarrow_{\text{CDCLChrono}}$ $(ML^{k+1}; N; U; \delta[L \mapsto k+1]; \top)$

provided $L$ is undefined in $M$ and $M$ is of level $k$

**Conflict** $(M; N; U; \delta; \top) \Rightarrow_{\text{CDCLChrono}} (M; N; U; \delta; D)$

provided $D \in (N \cup U)$ and $M \models \neg D$

**Skip** $(ML^{C\vee L}; N; U; \delta; D) \Rightarrow_{\text{CDCLChrono}}$
$(M; N; U; \delta[L \mapsto \infty]; D)$

provided $D \notin \{\top, \bot\}$ and $\mathrm{comp}(L)$ does not occur in $D$

**Resolve** $(ML^{C\vee L}; N; U; \delta; D \vee \mathrm{comp}(L)) \Rightarrow_{\text{CDCLChrono}}$
$(M; N; U; \delta[L \mapsto \infty]; D \vee C)$

provided $D$ and $L$ are of the same level

**Backtrack** $(M_1 K^k M_2; N; U; \delta; D \vee L) \Rightarrow_{\text{CDCLChrono}}$
$(M_1 M_3 L^{D\vee L}; N; U \cup \{D \vee L\}; \delta[M_4 \mapsto \infty][L \mapsto i]; \top)$

provided $L$ is of level $k$, $D$ is of level $i$, $M_3$ consists of all literals in $M_2$ of level $< k$, and $M_4$ consists of all literals in $M_2$ of level $\geq k$.

**Restart** $(M; N; U; \delta; \top) \Rightarrow_{\text{CDCLChrono}} (\epsilon; N; U; \delta_\infty; \top)$

provided $M \not\models N$

**Forget** $(M; N; U \uplus \{C\}; \delta; \top) \Rightarrow_{\text{CDCLChrono}} (M; N; U; \delta; \top)$

provided $M \not\models N$

Propagate ( $F$, $I$, $\delta$ )

**while** some $C \in F$ is unit $\{\ell\}$ under $I$ **do**
    $I := I\ell$
    $\delta(\ell) := \delta(C \setminus \{\ell\})$
    **for all** clauses $D \in F$ containing $\neg\ell$ **do**
        **if** $I(D) = \bot$ **then return** $D$
**return** $\bot$

Analyze ( $F$, $I$, $C$, $c$ )

**if** $C$ contains exactly one literal at decision level $c$ **then**
  $\ell :=$ literal in $C$ at decision level $c$
  $j := \delta(C \setminus \{\ell\})$
**else**
  $D :=$ *Learn* ( $I$, $C$ )
  $F := F \wedge D$
  $\ell :=$ literal in $D$ at decision level $c$
  $j := \delta(D \setminus \{\ell\})$
pick $b \in [\, j,\ c - 1\,]$
**for all** literals $k \in I$ with decision level $> b$ **do**
  assign $k$ decision level $\infty$
  remove $k$ from $I$
$I := I\ell$
assign $\ell$ decision level $j$