Organizatorial

Midterm: **8. December 14:00 – 16:00** (during the lecture slot)

Propositional Logic Calculi

1. ~~Tableau: classics, natural from the semantics~~
2. Resolution: classics, first-order, prepares for CDCL
3. CDCL: current prime calculus for propositional logic
4. Superposition: first-order, prepares for first-order

Resolution – quo vadis?

$$N = \{\underline{P \vee \overset{1}{Q}},\quad P \vee \neg Q,\quad \neg P \vee Q,\quad \underline{\neg P \vee \neg \overset{6}{Q}}\}$$

$N \Rightarrow_{RES}^{\text{Resolve}} N_1 = \{P \vee P\} \cup N$

$\Rightarrow_{RES}^{\text{Factoring}} N_2 = N_1 \cup \{P\}$

$\Rightarrow_{RES}^{\text{Resolve}} N_3 = N_2 \cup \{\neg P \vee \neg P\}$

$\Rightarrow_{RES}^{\text{Factoring}} N_4 = N_3 \cup \{\neg P\}$

$\Rightarrow_{RES}^{\text{Resolution}} N_4 \cup \{\bot\}$

But: $N \Rightarrow_{RES} N_1 = N \cup \{Q \vee \neg Q\}$

$\Rightarrow_{RES} N_1 \cup \{P \vee \neg P\}$

$\Rightarrow$ ...

CDCL – quo vadis?

$$N = \{P \lor Q, \quad P \lor \neg Q, \quad \neg P \lor Q, \quad \neg P \lor \neg Q\}$$

$(\varepsilon; N; \emptyset; 0; \top)$

$\Rightarrow_{\text{Decide}} (P^1; N; \emptyset; 1; \top)$

$\Rightarrow_{\text{Propagate}} (P^1 Q^{\neg P \lor Q}; N; \emptyset; 1; \top)$

$\Rightarrow_{\text{Conflict}} (P^1 Q^{\neg P \lor Q}; N; \emptyset; 1; \neg P \lor \neg Q)$

$\Rightarrow_{\text{Resolve}} (P^1; N; \emptyset; 1; \neg P \lor \neg P)$

$\Rightarrow_{\text{Factor, Backtrack}} (\neg P^{\neg P}; N; \{\neg P\}; 0; \top)$

Entering First-Order. . .

$$N = \{P(0),\quad \neg P(x) \lor P(S(x))\}$$

$\neg P(0) \lor P(S(0))$

Resolution:

$N \Rightarrow \{P(S(0))\} \Rightarrow \{P(S(S(0)))\} \Rightarrow \dots \Rightarrow \dots$

CDCL:

$\Rightarrow_{\text{Propagate}} (P(0)^{P(0)}; N; \emptyset; 0; \top)$

$\Rightarrow_{\text{Propagate}} (P(0)^{P(0)}\ P(S(0))^{\neg P(0) \lor P(S(0))}; \dots$

$\Rightarrow_{\text{Propagate}}$

$\vdots$

Propositional Superposition

Propositional Superposition refines the propositional resolution calculus by

(i) ordering and selection restrictions on inferences,

(ii) an abstract redundancy notion,

(iii) the notion of a partial model, based on the ordering for inference guidance

(iv) a *saturation* concept.

Important: No implicit Condensation of literals!

2.7.1 Definition (Clause Ordering)

Let $\prec$ be a total strict ordering on Σ.

Then $\prec$ can be lifted to a total ordering on literals by $\prec \subseteq \prec_L$ and $P \prec_L \neg P$ and $\neg P \prec_L Q$, $\neg P \prec_L \neg Q$ for all $P \prec Q$.

The ordering $\prec_L$ can be lifted to a total ordering on clauses $\prec_C$ by considering the multiset extension of $\prec_L$ for clauses.

$\Sigma = \{P, Q\}$

$P \prec Q$

$\neg P \succ_L P$ $\neg P \prec_L \neg Q$

$P \prec_L \neg P \prec_L Q \prec_L \neg Q$

$P \lor Q \prec_C P \lor P \lor Q$

$P \lor P \lor Q \prec_C P \lor Q \lor Q$

$P \lor Q \lor Q \prec_C \neg Q$

because $\{P, Q\} \prec_L^{mul} \{P, P, Q\} \prec_L^{mul} \{P, Q, Q\} \prec_L^{mul} \{\neg Q\}$

2.7.2 Proposition (Properties of the Clause Ordering)

(i) The orderings on literals and clauses are total and well-founded.

(ii) Let C and D be clauses with

$$P = \mathrm{atom}(\max(C))$$
$$Q = \mathrm{atom}(\max(D))$$

where $\max(C)$ denotes the maximal literal in C.

1. If $Q \prec_L P$ then $D \prec_C C$.
2. If $P = Q$, P occurs negatively in C but only positively in D, then $D \prec_C C$.

Eventually, I overload $\prec$ with $\prec_L$ and $\prec_C$.

For a clause set N, I define $N^{\prec C} = \{D \in N \mid D \prec C\}$.

- $\prec$ is an ordering on Σ
- $\prec_L$ is an extension for literals: $\prec \subseteq \prec_L$ and $P \prec_L \neg P$ and $\neg P \prec_L Q$, $\neg P \prec_L \neg Q$ for all $P \prec Q$.
- $\prec_C$ is the *multiset extension* of $\prec_L$ for clauses.

$$N^{\prec C} = \{D \in N \mid D \prec C\}$$

$N = \{P \vee \neg Q,\ P \vee Q,\ P \vee \neg S,\ P \vee Q \vee S\}$

$P < Q < R < S$ $\quad \prec_C$

redundant?

$C = P \vee Q \Rightarrow N^{<C} = \emptyset$ no!

$C = S \Rightarrow N^{<C} = \{P \vee Q,\ P \vee \neg Q\}$ no!

$C = \neg S \Rightarrow N^{<C} = \{P \vee Q,\ P \vee \neg Q,\ P \vee Q \vee S\}$ no!

$C = P \vee Q \vee S \Rightarrow N^{<C} = \{P \vee Q,\ P \vee \neg Q\}$ yes!

$(P \vee Q) \wedge (P \vee \neg Q) \models P \vee Q \vee S$

Definition (Abstract Redundancy)

A clause C is *redundant* with respect to a clause set N if $N^{\prec C} \models C$.

- Tautologies are always redundant.
- $N = \{ C_1, C_2, \ldots \}$

  $C_1 \subsetneq C_2 \Rightarrow C_2$ is redundant

  $P \vee Q$ $\quad$ $P \vee P \vee Q \vee Q$
- $N = \{ C_1, C_1, C_2, C_2, \ldots \} = \{ C_1, C_2, \ldots \}$

2.7.5 Definition (Selection Function)

The selection function sel maps clauses to one of its negative literals or $\perp$.

If $\mathrm{sel}(C) = \neg P$ then $\neg P$ is called *selected* in C.

If $\mathrm{sel}(C) = \perp$ then no literal in C is *selected*.

$P \vee \neg Q \vee \neg R \vee S$

2.7.6 Definition (Partial Model Construction)

Given a clause set N and an ordering $\prec$ we can construct a (partial) Herbrand model $N_{\mathcal{I}}$ for N inductively as follows:

$$N_C := \bigcup_{D \prec C, D \in N} \delta_D$$

positive!

$$\delta_D := \begin{cases} \{P\} & \text{if } D = D' \vee P,\ P \text{ strictly maximal, no literal} \\ & \text{selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases}$$

$$N_{\mathcal{I}} := \bigcup_{C \in N} \delta_C$$

Clauses C with $\delta_C \neq \emptyset$ are called *productive*.

- N: a *set of clauses*, interpreted as conjunction of all clauses.

$N = \{P \vee R, \neg P \vee Q, Q \vee S, Q \vee \neg R\}$

- $N_{\mathcal{I}}$, N_C are *sets of atoms*, often called *Herbrand Interpretations*.
- $N_{\mathcal{I}}$ is the overall (partial) model for N

$N_I = \{R, S\}$ (i)

  - $N_{\mathcal{I}} \models P$ if $P \in N_{\mathcal{I}}$
  - $N_{\mathcal{I}} \models \neg P$ if $P \notin N_{\mathcal{I}}$
  - $\mathcal{A}(N_{\mathcal{I}}) := N_{\mathcal{I}} \cup \{\neg P \mid P \notin N_{\mathcal{I}}\}$

$A(N_I) = \{R, S, \neg Q\}$ (ii)

$$\left.\begin{array}{l} A(N_I)(R) = 1 \\ A(N_I)(S) = 1 \\ A(N_I)(Q) = 0 \end{array}\right\} \text{(iii)}$$

$$N_C := \bigcup_{D \prec C, D \in N} \delta_D$$

$$\delta_D := \begin{cases} \{P\} & \text{if } D = D' \vee P, P \text{ strictly maximal, no literal} \\ & \text{selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases}$$

$N = \{ P \vee R^*, \ \neg P \vee Q^*, \ Q \vee S^*, \ Q \vee \neg R^* \}$

$P < Q < R < S$

$\neg P \vee Q^* < P \vee R^* < Q \vee \neg R^* < Q \vee S^*$

$$N_C := \bigcup_{D \prec C, D \in N} \delta_D$$

positive

$$\delta_D := \begin{cases} \{P\} & \text{if } D = D' \vee P, P \text{ strictly maximal, no literal selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases}$$

$$\neg P \vee Q^* \prec P \vee R^* \prec Q \vee \neg R^* \prec Q \vee S^*$$

clause D	N_D	δ_D	why?
$\neg P \vee Q^*$	$\emptyset$	$\emptyset$	$N_D \vdash D$, clause is true in N_D
$P \vee R^*$	$\emptyset$	$\{R\}$	$\{\neg P, \neg Q, \neg R, \neg S\} \not\models P \vee R$ ($= \emptyset$)
$Q \vee \neg R^*$	$\{R\}$	$\emptyset$	$\neg R^*$ is negative
$Q \vee S^*$	$\{R\}$	$\{S\}$	$Q \notin N_D$, $S \notin N_D \Rightarrow N_D \not\models Q \vee S$; S is strictly max.

$$N_I = \{R, S\}$$

$$N_I \not\models N$$

2.7.7 Proposition (Model Construction Properties)

Some properties of the partial model construction.

(i) For every D with $(C \lor \neg P) \prec D$ we have $\delta_D \neq \{P\}$.

(ii) If $\delta_C = \{P\}$ then $N_C \cup \delta_C \models C$.

(iii) If $N_C \models D$ and $D \prec C$ then for all C' with $C \prec C'$ we have $N_{C'} \models D$ and in particular $N_{\mathcal{I}} \models D$.

(iv) There is no clause C with $P \lor P \prec C$ such that $\delta_C = \{P\}$.

2.7.7 Proposition (Model Construction Properties)

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$$\delta_D \quad := \quad \begin{cases} \{P\} & \text{if } D = D' \vee P,\ P \text{ strictly maximal, no literal} \\ & \text{selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases}$$

2.7.7 Proposition (Model Construction Properties)

(ii) If $\delta_C = \{P\}$ then $N_C \cup \delta_C \models C$.

$$\delta_D \quad := \quad \begin{cases} \{P\} & \text{if } D = D' \vee P, P \text{ strictly maximal, no literal} \\ & \text{selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases}$$

2.7.7 Proposition (Model Construction Properties)

(i) For every D with $(C \vee \neg P) \prec D$ we have $\delta_D \neq \{P\}$.

(iii) If $N_C \models D$ and $D \prec C$ then for all C' with $C \prec C'$ we have $N_{C'} \models D$ and in particular $N_{\mathcal{I}} \models D$.

$$N_C \quad := \quad \bigcup_{D \prec C, D \in N} \delta_D$$

2.7.7 Proposition (Model Construction Properties)

(iv) There is no clause C with $P \vee P \prec C$ such that $\delta_C = \{P\}$.

$$\delta_D \quad := \quad \begin{cases} \{P\} & \text{if } D = D' \vee P,\ P \text{ strictly maximal, no literal} \\ & \text{selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases}$$

Superposition Inference Rules

"Resolution"

Superposition Left $(N \uplus \{C_1 \lor P, C_2 \lor \neg P\}) \Rightarrow_{\mathrm{SUP}}$
$(N \cup \{C_1 \lor P, C_2 \lor \neg P\} \cup \{C_1 \lor C_2\})$

where (i) P is strictly maximal in $C_1 \lor P$ (ii) no literal in $C_1 \lor P$ is selected (iii) $\neg P$ is maximal and no literal selected in $C_2 \lor \neg P$, or $\neg P$ is selected in $C_2 \lor \neg P$

"Factoring"

positive

Factoring $(N \uplus \{C \lor P \lor P\}) \Rightarrow_{\mathrm{SUP}}$
$(N \cup \{C \lor P \lor P\} \cup \{C \lor P\})$

where (i) P is maximal in $C \lor P \lor P$ (ii) no literal is selected in $C \lor P \lor P$

Superposition Left $(N \uplus \{C_1 \vee P, C_2 \vee \neg P\}) \Rightarrow_{\mathrm{SUP}}$ $(N \cup \{C_1 \vee P, C_2 \vee \neg P\} \cup \{C_1 \vee C_2\})$
where (i) P is strictly maximal in $C_1 \vee P$ (ii) no literal in $C_1 \vee P$ is selected (iii) $\neg P$ is maximal and no literal selected in $C_2 \vee \neg P$, or $\neg P$ is selected in $C_2 \vee \neg P$

$$\neg P \vee Q^* \prec P \vee R^* \prec Q \vee \neg R^* \prec Q \vee S^*$$

D	ω_D	δ_D	why?
$\neg P \vee Q^*$	$\emptyset$	$\emptyset$	
$P \vee R^*$	$\emptyset$	$\{R\}$	"productive counterpart"
$Q \vee \neg R^*$	$\{R\}$	$\emptyset$	"minimal false clause"
$Q \vee S^*$	$\{R\}$	$\{S\}$	

$N \Rightarrow_{\text{Superposition Left}} N \cup \{P \vee Q\}$

Superposition Left $(N \uplus \{C_1 \lor P, C_2 \lor \neg P\}) \Rightarrow_{\mathrm{SUP}}$ $(N \cup \{C_1 \lor P, C_2 \lor \neg P\} \cup \{C_1 \lor C_2\})$
where (i) P is strictly maximal in $C_1 \lor P$ (ii) no literal in $C_1 \lor P$ is selected (iii) $\neg P$ is maximal and no literal selected in $C_2 \lor \neg P$, or $\neg P$ is selected in $C_2 \lor \neg P$

$P \lor Q^* \prec \neg P \lor Q^* \prec P \lor R^* \prec Q \lor \neg R^* \prec Q \lor S^*$

D	N_D	δ_D	why?
$P \lor Q^*$	$\emptyset$	$\{Q\}$	Q^* is str. max, $N_D \not\models P \lor Q$
$\neg P \lor Q^*$	$\{Q\}$	$\emptyset$	true
$P \lor R^*$	$\{Q\}$	$\{R\}$	$N_D \not\models P \lor R$, R strict. max.
$Q \lor \neg R$	$\{Q, R\}$	$\emptyset$	true
$Q \lor S$	$\{Q, R\}$	$\emptyset$	true

$N_I = \{Q, R\}$ $N_I \models N$