interpretation of equality $\approx^{\mathcal{A}}$ is a congruence, Exercise **??**. Further on in this chapter I will also show that the other way round can hold as well: given a suitable congruence on some set, the equivalence classes of the congruence can then serve as the domain of a Σ-algebra providing a suitable interpretation for equality.

3.3 Substitutions

For a concrete propositional logic interpretation, it is sufficient select a valuation, i.e., truth values for the propositional variables, see Section 2.2. In first-order logic this becomes more versatile. The truth values for propositional variables correspond to n-ary relations on the domain with respect to valuations for the first-order variables, see Section 3.2. So in addition to the 0-relations for propositional variables, n-ary relations need to be considered under an assignment β for the first-order variables. When calculi for propositional logic considered partial interpretations, e.g., Tableau (Section 2.4) or CDCL (Section 2.9)), they are presented by sets of propositional literals taken from the processed clause set. For first-order logic this corresponds to taking first-order literals from the clause set and then instantiating the variables in these literals with terms in order to detect conflicts or for propagation. For example, a first-order clause $\neg P(x) \lor T(x)$ with universally quantified x propagates the literal $T(f(y))$ under the partial interpretation $P(f(y))$ where x is instantiated with $f(y)$. This instantiation is the syntactic counterpart of an assignment and represented by *substitutions* represented below.

Definition 3.3.1 (Substitution (well-sorted))**.** A *well-sorted substitution* is a mapping $\sigma : \mathcal{X} \to T(\Sigma, \mathcal{X})$ so that

1. $\sigma(x) \neq x$ for only finitely many variables x and
2. $\operatorname{sort}(x) = \operatorname{sort}(\sigma(x))$ for every variable $x \in \mathcal{X}$.

The application $\sigma(x)$ of a substitution σ to a variable x is often written in postfix notation as $x\sigma$. The variable set $\operatorname{dom}(\sigma) := \{x \in \mathcal{X} \mid x\sigma \neq x\}$ is called the *domain* of σ. The term set $\operatorname{codom}(\sigma) := \{x\sigma \mid x \in \operatorname{dom}(\sigma)\}$ is called the *codomain* of σ. From the above definition it follows that $\operatorname{dom}(\sigma)$ is finite for any substitution σ. The composition of two substitutions σ and τ is written as a juxtaposition $\sigma\tau$, i.e., $t\sigma\tau = (t\sigma)\tau$. A substitution σ is *more general* than a substitution τ if there is a substitution δ such that $\sigma\delta = \tau$ and we write $\sigma \leq \tau$. A substitution σ is called *idempotent* if $\sigma\sigma = \sigma$. A substitution σ is idempotent iff $\operatorname{dom}(\sigma) \cap \operatorname{vars}(\operatorname{codom}(\sigma)) = \emptyset$.

Substitutions are often written as sets of pairs $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ if $dom(\sigma) = \{x_1, \ldots, x_n\}$ and $x_i\sigma = t_i$ for every $i \in \{1, \ldots, n\}$. The *modification* of a substitution σ at a variable x is defined as follows:

$$\sigma[x \mapsto t](y) = \begin{cases} t & \text{if } y = x \\ \sigma(y) & \text{otherwise} \end{cases}$$

A substitution σ is identified with its extension to formulas and defined as follows:

1. $\bot\sigma = \bot$,
2. $\top\sigma = \top$,
3. $(f(t_1, \dots, t_n))\sigma = f(t_1\sigma, \dots, t_n\sigma)$,
4. $(P(t_1, \dots, t_n))\sigma = P(t_1\sigma, \dots, t_n\sigma)$,
5. $(s \approx t)\sigma = (s\sigma \approx t\sigma)$,
6. $(\neg\phi)\sigma = \neg(\phi\sigma)$,
7. $(\phi \circ \psi)\sigma = \phi\sigma \circ \psi\sigma$ where $\circ \in \{\vee, \wedge\}$,
8. $(Qx.\phi)\sigma = Qz.(\phi\sigma[x \mapsto z])$ where $Q \in \{\forall, \exists\}$, z and x are of the same sort and z is a fresh variable.

The result $t\sigma$ ($\phi\sigma$) of applying a substitution σ to a term t (formula ϕ) is called an *instance* of t (ϕ). The substitution σ is called *ground* if it maps every domain variable to a ground term, i.e., the codomain of σ consists of ground terms only. If the application of a substitution σ to a term t (formula ϕ) produces a ground term $t\sigma$ (a variable-free formula, $\text{vars}(\phi\sigma) = \emptyset$), then $t\sigma$ ($\phi\sigma$) is called *ground instance* of t (ϕ) and σ is called *grounding* for t (ϕ). The set of ground instances of a clause set N is given by $\text{grd}(\Sigma, N) = \{C\sigma \mid C \in N, \sigma \text{ is grounding for } C\}$ is the set of *ground instances* of N. A substitution σ is called a *variable renaming* if $\text{codom}(\sigma) \subseteq \mathcal{X}$ and for any $x, y \in \mathcal{X}$, if $x \neq y$ then $x\sigma \neq y\sigma$, i.e., σ is a bijection $\mathcal{X}$ into $\mathcal{X}$.

The following lemma establishes the relationship between substitutions and assignments.

Lemma 3.3.2 (Substitutions and Assignments)**.** Let β be an assignment of some interpretation $\mathcal{A}$ of a term t and σ a substitution. Then

$$\beta(t\sigma) = \beta[x_1 \mapsto \beta(x_1\sigma), \dots, x_n \mapsto \beta(x_n\sigma)](t)$$

where $\text{dom}(\sigma) = \{x_1, \dots, x_n\}$.

Proof. By structural induction on t. If $t = a$ is a constant, then $\beta(a\sigma) = a^{\mathcal{A}} = \beta[x_1 \mapsto \beta(x_1\sigma), \dots, x_n \mapsto \beta(x_n\sigma)](a)$. The case $t = x$ is a variable and $x \notin \text{dom}(\sigma)$ is identical to the case that t is a constant. So $t = x_i$ is a variable and $x_i \in \text{dom}(\sigma)$, where $x_i\sigma = s$. If s is a variable, then $\beta(t\sigma) = \beta(x_i\sigma) = \beta(s) = \beta[x_i \mapsto \beta(s)](x_i) = \beta[x_1 \mapsto \beta(x_1\sigma), \dots, x_n \mapsto \beta(x_n\sigma)](t)$. The case s is a constant is analogous to the case t is a constant. So let $x_i\sigma = s = f(s_1, \dots, s_m)$. $\beta(x_i\sigma) = \beta(f(s_1, \dots, s_m)) = f^{\mathcal{A}}(\beta(s_1), \dots, \beta(s_m)) = \beta[x_i \mapsto f(s_1, \dots, s_m)](x_i) = \beta[x_1 \mapsto \beta(x_1\sigma), \dots, x_n \mapsto \beta(x_n\sigma)](t)$.
For the inductive case let $t = f(t_1, \dots, t_m)$. Then $\beta(t\sigma) = f^{\mathcal{A}}(\beta(t_1\sigma), \dots, \beta(t_m\sigma)) = f^{\mathcal{A}}(\beta[x_1 \mapsto \beta(x_1\sigma), \dots, x_n \mapsto \beta(x_n\sigma)](t_1), \dots, \beta[x_1 \mapsto \beta(x_1\sigma), \dots, x_n \mapsto \beta(x_n\sigma)](t_m)) = \beta[x_1 \mapsto \beta(x_1\sigma), \dots, x_n \mapsto \beta(x_n\sigma)](t)$. □

Corollary 3.3.3. Let ϕ be a quantifier free formula, σ a substitution, $\mathcal{A}$ be a Σ-algebra, and β an assignment. Then $\mathcal{A}, \beta \models \phi\sigma$ iff $\mathcal{A}, \beta[x_1 \mapsto \beta(x_1\sigma), \ldots, x_n \mapsto \beta(x_n\sigma)] \models \phi$, where $\mathrm{dom}(\sigma) = \{x_1, \ldots, x_n\}$.

3.4 Equality

The equality predicate is build into the first-order language in Section 3.1 and not part of the signature. It is a first class citizen. This is the case although it can be actually axiomatized in the language. The motivation is that firstly, many real world problems naturally contain equations. They are a means to define functions. Then predicates over terms model properties of the functions. Secondly, without special treatment in a calculus, it is almost impossible to automatically prove non-trivial properties of a formula containing equations.

In this section I firstly show that any formula can be transformed into a formula where all atoms are equations. Secondly, that any formula containing equations can be transformed into a formula where the equality predicate is replaced by a fresh predicate together with some axioms. In the first case the respective clause sets are equivalent, in the second case the transformation is satisfiability preserving. For the replacement of any predicate R by equations over a fresh function f_R we assume an additional fresh sort Bool with a fresh constant true.

InjEq $\chi[R(t_{1,1}, \ldots, t_{1,n})]_{p_1} \ldots [R(t_{m,1}, \ldots, t_{m,n})]_{p_m} \Rightarrow_{\mathrm{IE}} \chi[f_R(t_{1,1}, \ldots, t_{1,n}) \approx \mathrm{true}]_{p_1} \ldots [f_R(t_{m,1}, \ldots, t_{m,n}) \approx \mathrm{true}]_{p_m}$

provided R is a predicate occurring in χ, $\{p_1, \ldots, p_m\}$ are all positions of atoms with predicate R in χ and f_R is new with appropriate sorting

Theorem 3.4.1. Let $\chi \Rightarrow^*_{\mathrm{IE}} \chi'$ then χ is satisfiable (valid) iff χ' is satisfiable (valid).

Proof. (Sketch) The basic proof idea is to establish the relation $(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) \in R^{\mathcal{A}}$ iff $f_R^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) = \mathrm{true}^{\mathcal{A}}$. Furthermore, the sort of true is fresh to χ and the equations $f_R(t_1, \ldots, t_n) \approx \mathrm{true}$ do not interfere with any term t_i because the f_R are all fresh and only occur on top level of the equations. □

When removing equality from a formula it needs to be axiomatized. For simplicity, I assume here that the considered formula χ is one-sorted, i.e., there is only one sort occurring for functions, relations in χ. The extension to formulas with many sorts is straightforward and discussed below.

RemEq $\chi[l_1 \approx r_1]_{p_1} \ldots [l_m \approx r_m]_{p_m} \Rightarrow_{\mathrm{RE}} \chi[E(l_1, r_1)]_{p_1} \ldots [E(l_m, r_m)]_{p_m} \wedge \mathrm{def}(\chi, E)$

provided $\{p_1, \ldots, p_m\}$ are all positions of equations $l_i \approx r_i$ in χ and E is a new binary predicate

The formula $\text{def}(\chi, E)$ is the axiomatization of equality for χ and it consists of a conjunction of the equivalence relation axioms for E

$\forall x.E(x,x)$
$\forall x,y.(E(x,y) \to E(y,x))$
$\forall x,y,z.((E(x,y) \wedge E(x,z)) \to E(x,z))$

plus the congruence axioms for E for every n-ary function symbol f

$\forall x_1, y_1, \ldots, x_n, y_n.((E(x_1,y_1) \wedge \ldots \wedge E(x_n,y_n))$
$\quad \to E(f(x_1,\ldots,x_n), f(y_1,\ldots,y_n)))$

plus the congruence axioms for E for every m-ary predicate symbol P

$\forall x_1, y_1, \ldots, x_m, y_m.((E(x_1,y_1) \wedge \ldots \wedge E(x_m,y_m) \wedge P(x_1,\ldots,x_m))$
$\quad \to P(y_1,\ldots,y_m)$

Theorem 3.4.2. Let $\chi \Rightarrow_{\text{RE}} \chi'$ then χ is satisfiable iff χ' is satisfiable.

Proof. ($\Rightarrow$) Let $\mathcal{A}$ be an algebra with $\mathcal{A} \models \chi$. Now extend $\mathcal{A}$ to an interpretation $\mathcal{A}'$ for χ' by assigning $E^{\mathcal{A}'} :=\approx^{\mathcal{A}}$ and $\mathcal{A}' := \mathcal{A}$ otherwise. Obviously, $\mathcal{A}' \models \chi'$ because $\approx$ satisfies the equivalence and congruence axioms and in χ' and $(a,b) \in E^{\mathcal{A}'}$ iff $a \approx b$.

($\Leftarrow$) Assume $\mathcal{A}'$ is a model for χ'. Now define a relation $\sim$ by $a :\sim b$ for $a, b \in \mathcal{U}^{\mathcal{A}'}$ if $(a,b) \in E^{\mathcal{A}'}$. The equivalence axioms for E are part of χ' so we can define equivalence classes $[a] := \{b \mod a \sim b\}$ for all $a, b \in \mathcal{U}^{\mathcal{A}'}$. Obviously, the definition of equlivalence classes respects the sort restrictions. Next we define the domain of $\mathcal{A}$ to be $\mathcal{U}^{\mathcal{A}} := \{[a] \mid a \in \mathcal{U}^{\mathcal{A}'}\}$. We interpret functions in $\mathcal{A}$ by the usual homomorphism $f^{\mathcal{A}}([a_1],\ldots,[a_n]) := [f^{\mathcal{A}'}(a_1,\ldots,a_n)]$ and relations by $R^{\mathcal{A}}([a_1],\ldots,[a_n]) := R^{\mathcal{A}'}(a_1,\ldots,a_n)$. Both definitions are well-defined because of the congruence axioms for any f, R in χ'. Then we get $[a] \approx [b]$ in $\mathcal{A}$ iff $[a] = [b]$ in $sigval'$ iff $a \sim b$ in $sigval'$. Now, $\mathcal{A} \models \chi$ can be shown buy strucural induction on χ where I only show the two relevant cases for terms and atoms for $\mathcal{A}, \beta$ and $\mathcal{A}', \beta'$, respectively. As an invariant through the case of quantifiers, I assume $\beta(x) = [b]$ iff $\beta'(x) = b$ for any variable x.
Firstly, $f(t_1,\ldots,t_n)^{\mathcal{A},\beta} = [a]$ iff $f(t_1,\ldots,t_n)^{\mathcal{A}',\beta'} = a$ by structural induction.
Secondly, $\mathcal{A}, \beta \models R(t_1,\ldots,t_n)$ by definition if $(t_1,\ldots,t_n)^{\mathcal{A},\beta} \in R^{\mathcal{A}}$ which is the case if $(t_1,\ldots,t_n)^{\mathcal{A}',\beta'} \in R^{\mathcal{A}'}$. ☐

Corollary 3.4.3. Let $\chi \Rightarrow_{\text{RE}} \chi'$ where $\chi' = \chi[E(l_1,r_1)]_{p_1} \ldots [E(l_m,r_m)]_{p_m} \wedge \text{def}(\chi, E)$. Then $\models \chi$ iff $\models \text{def}(\chi,E) \to \chi[E(l_1,r_1)]_{p_1} \ldots [E(l_m,r_m)]_{p_m}$.

Proof. It holds $\models \chi$ iff $\neg\chi$ is unsatisfiable iff $(\neg\chi)[E(l_1,r_1)]_{1p_1} \ldots [E(l_m,r_m)]_{1p_m} \wedge \text{def}(\chi,E)$ is unsatisfiable, by Theorem 3.4.2 and the definition of $\Rightarrow_{\text{RE}}$, iff $\models \text{def}(\chi,E) \to \chi[E(l_1,r_1)]_{p_1} \ldots [E(l_m,r_m)]_{p_m}$. ☐

Now in case χ has many different sorts then for each sort S one new fresh predicate E_S is needed for the translation. For each of these predicates equivalence relation and congruence axioms need to be generated where for every function f only one axiom using E_S is needed, where S is the range sort of S. Similar for the domain sorts of f and accordingly for predicates.

3.5 Herbrand's Theorem

There are substantial differences between propositional logic and its generalization first-order logic. There are only finitely many formulas in propositional logic that can be semantically distinguished for some finite signature. Given a finite propositional signature Σ there are "only" $2^{|\Sigma|}$ different valuations. In first-order logic there are infinitely many different interpretations for formulas over some finite first-order signature Σ. As we will see, this moves the satisfiability problem for some set of clauses from NP (propositional) to undecidable (first-order), see Section 3.15. In this section I present two results that are the basis for most first-order calculi. Firstly, I show that when considering satisfiability of a clause set, it is not necessary to consider arbitrary interpretations. Instead, one specific interpretation, called *Herbrand interpretation*, is sufficient for establishing satisfiability. Secondly, interpretations for first-order clause sets, including Herbrand interpretations, typically consider an infinite domain. This implies infinitely many different assignments defining the semantics for a clause set. Still, if some clause set is unsatisfiable, then finitely many assignments are sufficient to prove unsatisfiability. This property is called *Compactness* of first-order logic. Putting the two results together, it is sufficient to consider finitely many assignments from the Herbrand interpretation in order to prove unsatisfiability of a set of clauses: the basis for all modern automated reasoning calculi for first-order logic.

Definition 3.5.1 (Herbrand Interpretation)**.** A *Herbrand Interpretation* (over Σ) is a Σ-algebra $\mathcal{H}$ such that

1. $S^{\mathcal{H}} := T_S(\Sigma)$ for every sort $S \in \mathcal{S}$
2. $f^{\mathcal{H}} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n)$ where $f \in \Omega$, $\text{arity}(f) = n$, $s_i \in S_i^{\mathcal{H}}$ and $f : S_1 \times \ldots \times S_n \to S$ is the sort declaration for f
3. $P^{\mathcal{H}} \subseteq (S_1^{\mathcal{H}} \times \ldots \times S_m^{\mathcal{H}})$ where $P \in \Pi$, $\text{arity}(P) = m$ and $P \subseteq S_1 \times \ldots \times S_m$ is the sort declaration for P

Lemma 3.5.2 (Herbrand Interpretations are Well-Defined)**.** Every Herbrand Interpretation is a Σ-algebra.

Proof. (i) the carriers are non-empty because every signature contains a constant declaration for each sort. If $S^{\mathcal{H}} \cap T^{\mathcal{H}} \neq \emptyset$, then there must be two declarations for the same function symbol in Σ which is forbidden. Furthermore, $\sim$ is well-sorted.

(ii) functions are total by definition.

(iii) relations are assigned. □

In other words, values for ground terms are fixed to be the ground terms itself and functions are fixed to be the term constructors. Predicate symbols may be freely interpreted as relations over ground terms.

Proposition 3.5.3 (Representing Herbrand Interpretations)**.** A Herbrand interpretation $\mathcal{A}$ can be uniquely determined by a set of ground atoms I

$$(s_1, \ldots, s_n) \in P^{\mathcal{A}} \ \text{ iff } \ P(s_1, \ldots, s_n) \in I$$

Thus Herbrand interpretations (over Σ) can be identified with sets of Σ-ground atoms. A Herbrand interpretation I is called a *Herbrand model* of ϕ, where I assume ϕ does not contain equations, if $I \models \phi$.

C

Historically, Herbrand interpretations have been defined for first-order logic without equality. These are exactly the definitions above. Later on, I'll extend these notions such that they also cover the case of equations.

Example 3.5.4. Consider the signature $\Sigma = (\{S\}, \{a, b\}, \{P, Q\})$, where a, b are constants, $\text{arity}(P) = 1$, $\text{arity}(Q) = 2$, and all constants, predicates are defined over the sort S. Then the following are examples of Herbrand interpretations over Σ, where for all interpretations $S_{\mathcal{A}} = \{a, b\}$.

$I_1 \,:\, = \emptyset$
$I_2 \,:\, = \{P(a), Q(a, a), Q(b, b)\}$
$I_3 \,:\, = \{P(a), P(b), Q(a, a), Q(b, b), Q(a, b), Q(b, a)\}$

Now consider the extension Σ' of Σ by one unary function symbol $g : S \to S$. Then the following are examples of Herbrand interpretations over Σ', where for all interpretations $S_{\mathcal{A}} = \{a, b, g(a), g(b), g(g(a)), \ldots\}$.

$I'_1 \,:\, = \emptyset$
$I'_2 \,:\, = \{P(a), Q(a, g(a)), Q(b, b)\}$
$I'_3 \,:\, = \{P(a), P(g(a)), P(g(g(a))), \ldots, Q(a, a), Q(b, b), Q(b, g(b)), Q(b, g(g(b))), \ldots\}$

Theorem 3.5.5 (Herbrand's Theorem)**.** Let N be a finite set of Σ-clauses without equality. Then N is satisfiable iff N has a Herbrand model over Σ iff $\text{grd}(\Sigma, N)$ has a Herbrand model over Σ.

Proof. Firstly, I prove that if N has a model, then it has a Herbrand model over Σ. So let $\mathcal{A}$ be a model for N. Since N is finite let's consider exactly the subsignature of N. Then $P^{\mathcal{H}} = \{(t_1, \ldots, t_n) \mid (t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) \in P^{\mathcal{A}}, t_i \in T(\Sigma)\}$. Finally, I need to prove that $\mathcal{H}$ is a model for N. Assume not. Then there is a clause $C \in N$ and an assignment $\beta_{\mathcal{H}}$ such that $\mathcal{H}(\beta_{\mathcal{H}})(C) = 0$ where $\beta_{\mathcal{H}}(x_i) = t_i$ for all $x_i \in \text{vars}(C)$ with $t_i \in T_{\text{sort}(x_i)}(\Sigma)$. Let $\sigma = \{x_1 \mapsto t_1, \ldots, x_m \mapsto t_m\}$. Now consider an assignment $\beta_{\mathcal{A}}$ where $\beta_{\mathcal{A}}(x_i) = t_i^{\mathcal{A}}$. Since $\mathcal{A} \models N$ also $\mathcal{A}(\beta_{\mathcal{A}}) \models C$, in particular, there is a literal $L \in C$ with $\mathcal{A}(\beta_{\mathcal{A}})(L) = 1$. If it is an atom $P(l_1, \ldots, l_n)$ with $(\mathcal{A}(\beta_{\mathcal{A}})(l_1), \ldots, \mathcal{A}(\beta_{\mathcal{A}})(l_n)) \in P^{\mathcal{A}}$, but then $(l_1\sigma, \ldots, l_n\sigma) \in P^{\mathcal{H}}$ by definition of $\mathcal{H}$ and Lemma 3.3.2. Hence $(\mathcal{H}(\beta_{\mathcal{H}})(l_1), \ldots, \mathcal{H}(\beta_{\mathcal{H}})(l_n)) \in P^{\mathcal{H}}$, a contradiction. The case where L is negative is dual.
Secondly, due to Lemma 3.5.2 the existence of a Herbrand model implies satisfiability.
It remains to be shown that N has a Herbrand model over Σ iff $\text{grd}(\Sigma, N)$ has a Herbrand model. Firstly, assume N has a Herbrand model $\mathcal{H}$ over Σ. Then $\mathcal{H}$ is

also a model for $\operatorname{grd}(\Sigma, N)$. Assume not. Then there is a clause $C\sigma \in \operatorname{grd}(\Sigma, N)$, $C \in N$, such that $\mathcal{H} \not\models C\sigma$. But then $\mathcal{H}(\beta_{\mathcal{H}}[x_1 \mapsto (x_1\sigma), \ldots, x_n \mapsto (x_n\sigma)])(C) = 0$, $\operatorname{dom}(\sigma) = \{x_1, \ldots, x_n\}$, contradicting $\mathcal{H}$ is a model for N. Secondly, assume $\mathcal{H}$ is a model for $\operatorname{grd}(\Sigma, N)$. Then $\mathcal{H}$ is also a model for N. Assume not, then there is a clause $C \in N$ and ad assignment $\beta_{\mathcal{H}}[x_1 \mapsto (x_1\sigma), \ldots, x_n \mapsto (x_n\sigma)]$, $vars(C) = \{x_1, \ldots, x_n\}$, such that $\mathcal{H}(\beta_{\mathcal{H}}[x_1 \mapsto (x_1\sigma), \ldots, x_n \mapsto (x_n\sigma)])(C) = 0$. But then $\mathcal{H} \not\models C\sigma$, contradicting $\mathcal{H}$ is a model for $\operatorname{grd}(\Sigma, N)$.

□

Example 3.5.6 (Example of a $\operatorname{grd}(\Sigma, N)$)**.** Consider Σ' from Example 3.5.4 and the clause set $N = \{Q(x,x) \lor \neg P(x), \neg P(x) \lor P(g(x))\}$. Then the set of ground instances $\operatorname{grd}(\Sigma', N) = \{$

$Q(a,a) \lor \neg P(a)$
$Q(b,b) \lor \neg P(b)$
$Q(g(a),g(a)) \lor \neg P(g(a))$
$\ldots$
$\neg P(a) \lor P(g(a))$
$\neg P(b) \lor P(g(b))$
$\neg P(g(a)) \lor P(g(g(a)))$
$\ldots\}$

is satisfiable. For example by the Herbrand models

$I_1 \;:\; = \emptyset$
$I_2 \;:\; = \{P(b), Q(b,b), P(g(b)), Q(g(b), g(b)), \ldots\}$

Definition 3.5.7 (Herbrand Interpretation with Equality)**.** A *Herbrand Interpretation* (over Σ) is a Σ-algebra $\mathcal{H}$ such that

1. a well-sorted equivalence relation $\sim$ on $T(\Sigma)$, i.e., if $s \sim t$ then $s, t \in T_S(\Sigma)$ for some S where $[s]$ denotes the equivalence class containing s

2. $S^{\mathcal{H}} := T_S(\Sigma)/\sim$ for every sort $S \in \mathcal{S}$

3. $f^{\mathcal{H}} : ([s_1], \ldots, [s_n]) \mapsto [f(s_1, \ldots, s_n)]$ where $f \in \Omega$, $\operatorname{arity}(f) = n$, $s_i \in T_{S_i}(\Sigma)$ and $f : S_1 \times \ldots \times S_n \to S$ is the sort declaration for f

4. $P^{\mathcal{H}} \subseteq (S_1^{\mathcal{H}} \times \ldots \times S_m^{\mathcal{H}})$ where $P \in \Pi$, $\operatorname{arity}(P) = m$ and $P \subseteq S_1 \times \ldots \times S_m$ is the sort declaration for P

Lemma 3.5.8 (Herbrand Interpretations are Well-Defined)**.** Every Herbrand Interpretation is a Σ-algebra.

Proof. (i) the carriers are non-empty because every signature contains a constant declaration for each sort. If $S^{\mathcal{H}} \cap T^{\mathcal{H}} \neq \emptyset$, then there must be two declarations for the same function symbol in Σ which is forbidden. Furthermore, $\sim$ is well-sorted.

(ii) functions are total by definition.

(iii) relations are assigned. □