First-Order Tableau

The different versions of tableau for first-order logic differ in particular in the treatment of variables by the tableau rules. The first variant I consider is standard first-order tableau where variables are instantiated by ground terms. For this section, if not stated otherwise, all considered formulas are closed and do not contain equations.

3.6.1 Definition (γ-,δ-Formulas)

A formula ϕ is called a γ-formula if ϕ is a formula $\forall x_S.\psi$ or $\neg\exists x_S.\psi$.

A formula ϕ is called a δ-formula if ϕ is a formula $\exists x_S.\psi$ or $\neg\forall x_S.\psi$.

3.6.2 Definition (Direct Standard Tableau Descendant)

Given a γ- or δ-formula ϕ, its direct descendants are:

γ	Descendant $\gamma(t)$
$\forall x_S.\psi$	$\psi\{x_S \mapsto t\}$
$\neg\exists x_S.\psi$	$\neg\psi\{x_S \mapsto t\}$
	for a ground term $t \in T_S(\Sigma)$

δ	Descendant $\delta(c)$
$\exists x_S.\psi$	$\psi\{x_S \mapsto c\}$
$\neg\forall x_S.\psi$	$\neg\psi\{x_S \mapsto c\}$
	for a fresh constant $c \in T_S(\Sigma)$

A sequence of formulas $(\phi_1, \phi_2, \ldots)$ is called *closed* if there are two formulas ϕ_i and ϕ_j in the sequence where $\phi_i = \text{comp}(\phi_j)$.

A state is *closed* if all its formula sequences are closed.

Given a formula ϕ, the initial tableau for ϕ is $\{((\neg\phi), \{c_1, \ldots, c_k\})\}$ where the c_i are constants for each sort S such that ϕ contains a variable of sort S but no ground term t in ϕ has sort S.

A first-order formula ϕ occurring in some sequence in *N* of a pair (M, J) is called *open*

if in case ϕ is an α-formula not both direct descendants are already part of *M*,

if it is a β-formula none of its descendants is part of *M*,

if it is a δ-formula no direct descendant is part of *M*, and

if it is a γ-formula not all direct descendants are part of *M* where only ground terms are considered that can be built from constants *J* and function symbols in *M* such that their depth is bounded by the maximal term depth occurring in *M*.

Tableau Rules

α**-Expansion** $N \uplus \{((\phi_1, \ldots, \psi, \ldots, \phi_n), J)\} \Rightarrow_{FT}$
$N \uplus \{((\phi_1, \ldots, \psi, \ldots, \phi_n, \psi_1, \psi_2), J)\}$
provided ψ is an open α-formula, ψ_1, ψ_2 its direct descendants and the sequence is not closed.

β**-Expansion** $N \uplus \{((\phi_1, \ldots, \psi, \ldots, \phi_n), J)\} \Rightarrow_{FT}$
$N \uplus \{((\phi_1, \ldots, \psi, \ldots, \phi_n, \psi_1), J)\} \uplus \{((\phi_1, \ldots, \psi, \ldots, \phi_n, \psi_2), J)\}$
provided ψ is an open β-formula, ψ_1, ψ_2 its direct descendants and the sequence is not closed.

γ-Expansion $N \uplus \{((\phi_1, \ldots, \psi, \ldots, \phi_n), J)\} \Rightarrow_{\mathrm{FT}}$
$N \uplus \{((\phi_1, \ldots, \psi, \ldots, \phi_n, \psi'), J)\}$
provided ψ is a γ-formula, ψ' a $\gamma(t)$ descendant where t is a ground term in the signature of the sequence and J, the depth of t is bounded by the maximal term depth in $(\phi_1, \ldots, \psi, \ldots, \phi_n)$ and the sequence is not closed.

δ-Expansion $N \uplus \{((\phi_1, \ldots, \psi, \ldots, \phi_n), J)\} \Rightarrow_{\mathrm{FT}}$
$N \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n, \psi'), J \cup \{c\})\}$
provided ψ is an open δ-formula, ψ' a $\delta(c)$ descendant where c is a fresh constant and the sequence is not closed.

The α-, and β-expansion rules are copies of the propositional rules adjusted to the extended state format. The α-, and β-expansion rules don't manipulate the set of ground terms of a state.

Note that for a particular γ-Expansion the number of potential ground terms that can be used for instantiation is always finite.

A possibly infinite tableau derivation $s_0 \Rightarrow_{FT} s_1 \Rightarrow_{FT} \dots$ is called *saturated* if for all its open sequences M_i of some pair $(M_i, J_i) \in s_i$ and all formulas ϕ occurring in M_i, there is an index $j > i$ and some pair $(M_j, J_j) \in s_j$, such that M_i is a prefix of M_j:

if in case ϕ is an α-formula then both direct descendants are part of M_j,

if it is a β-formula then one of its descendants is part of M_j,

if it is a δ-formula then one direct descendant is part of M_j , and

if it is γ-formula then all direct descendants from ground terms that can be built from function symbols in J_i and M_i such that their depth is bounded by the maximal term depth in M_i are part of the sequence M_j.

3.6.5 Theorem (Standard First-Order Tableau is Sound)

If for a closed formula ϕ the tableau calculus derives $\{((\neg\phi), J)\} \Rightarrow^{*}_{\mathrm{FT}} N$ and N is closed, then ϕ is valid.

3.6.6 Definition (Hintikka Set)

Let M be a possibly infinite set of closed first-order formulas such that for each variable x_S occurring in M, there is at least on ground term t of sort S occurring in M. M is a *Hintikka set* if for all formulas $\phi \in M$:

1. if ϕ is a literal, then comp(ϕ) $\notin M$
2. if ϕ is an α formula then α_1 and α_2 are in M
3. if ϕ is a β formula then β_1 or β_2 is in M
4. if ϕ is a γ formula then for all ground terms t of correct sort occurring in M the formula $\gamma(t)$ is in M
5. if ϕ is a δ formula then $\delta(c)$ is in M for some constant c

3.6.7 Lemma (Hintikka's Lemma)

A Hintikka set is satisfiable.

3.6.8 Lemma (Open Branches of Saturated Tableaux constitute Hintikka Sets)

Let $s_0 \Rightarrow_{\text{FT}} s_1 \Rightarrow_{\text{FT}} \ldots$ be a saturated tableau derivation containing the derivation of an open branch $(M_1, J_1) \Rightarrow (M_2, J_2) \Rightarrow \ldots$ where M_i is always a strict subsequence of M_{i+1}. Then $M = \bigcup_i M_i$ is a Hintikka set.

3.6.9 Theorem (Standard First-Order Tableau is Complete)

If ϕ is valid then the tableau calculus computes $\{((\neg\phi), J)\} \Rightarrow^*_{\text{FT}} N$ and N is closed.