Polynomial Unification

The second calculus, polynomial unification, prevents the problem of exponential growth by introducing an implicit representation for the mgu.

For this calculus the size of a normal form is always polynomial in the size of the input unification problem.

Tautology $E \uplus \{t = t\} \;\Rightarrow_{\mathrm{PU}}\; E$

Decomposition $E \uplus \{f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)\} \;\Rightarrow_{\mathrm{PU}}$
$E \uplus \{s_1 = t_1, \ldots, s_n = t_n\}$

Clash $E \uplus \{f(t_1, \ldots, t_n) = g(s_1, \ldots, s_m)\} \;\Rightarrow_{\mathrm{PU}}\; \bot$
if $f \neq g$

Occurs Check $E \uplus \{x = t\} \Rightarrow_{\mathrm{PU}} \bot$

if $x \neq t$ and $x \in \mathit{vars}(t)$

Orient $E \uplus \{t = x\} \Rightarrow_{\mathrm{PU}} E \uplus \{x = t\}$

if $t \notin \mathcal{X}$

Substitution $E \uplus \{x = y\} \Rightarrow_{\mathrm{PU}} E\{x \mapsto y\} \uplus \{x = y\}$

if $x \in \mathit{vars}(E)$ and $x \neq y$

Cycle $E \uplus \{x_1 = t_1, \ldots, x_n = t_n\} \Rightarrow_{\mathrm{PU}} \bot$

if there are positions p_i with $t_i|_{p_i} = x_{i+1}, t_n|_{p_n} = x_1$ and some $p_i \neq \epsilon$

Merge $E \uplus \{x = t, x = s\} \Rightarrow_{\mathrm{PU}} E \uplus \{x = t, t = s\}$

if $t, s \notin \mathcal{X}$ and $|t| \leq |s|$

3.7.4 Theorem (Soundness, Completeness and Termination of $\Rightarrow_{\mathrm{PU}}$)

If s, t are two terms with $\mathrm{sort}(s) = \mathrm{sort}(t)$ then

1. if $\{s = t\} \Rightarrow^*_{\mathrm{PU}} E$ then any equation $(s' = t') \in E$ is well-sorted, i.e., $\mathrm{sort}(s') = \mathrm{sort}(t')$.
2. $\Rightarrow_{\mathrm{PU}}$ terminates on $\{s = t\}$.
3. if $\{s = t\} \Rightarrow^*_{\mathrm{PU}} E$ then σ is a unifier (mgu) of E iff σ is a unifier (mgu) of $\{s = t\}$.
4. if $\{s = t\} \Rightarrow^*_{\mathrm{PU}} \bot$ then s and t are not unifiable.

3.7.5 Theorem (Normal Forms Generated by $\Rightarrow_{\mathrm{PU}}$)

Let $\{s = t\} \Rightarrow^*_{\mathrm{PU}} \{x_1 = t_1, \ldots, x_n = t_n\}$ be a normal form. Then

1. $x_i \neq x_j$ for all $i \neq j$ and without loss of generality $x_i \notin \mathrm{vars}(t_{i+k})$ for all i, k, $1 \leq i < n$, $i + k \leq n$.
2. the substitution $\{x_1 \mapsto t_1\}\{x_2 \mapsto t_2\} \ldots \{x_n \mapsto t_n\}$ is an mgu of $s = t$.