Unification

3.7.1 Definition (Unifier)

Two terms s and t of the same sort are said to be *unifiable* if there exists a well-sorted substitution σ so that $s\sigma = t\sigma$, the substitution σ is then called a well-sorted *unifier* of s and t.

The unifier σ is called *most general unifier*, written $\sigma = mgu(s, t)$, if any other well-sorted unifier τ of s and t it can be represented as $\tau = \sigma\tau'$, for some well-sorted substitution τ'.

A state of the naive standard unification calculus is a set of equations E or $\bot$, where $\bot$ denotes that no unifier exists. The set E is also called a *unification problem*.

The start state for checking whether two terms s, t, $\text{sort}(s) = \text{sort}(t)$, (or two non-equational atoms A, B) are unifiable is the set $E = \{s = t\}$ ($E = \{A = B\}$). A variable x is *solved* in E if $E = \{x = t\} \uplus E'$, $x \not\in \text{vars}(t)$ and $x \not\in \text{vars}(E)$.

A variable $x \in \text{vars}(E)$ is called *solved* in E if $E = E' \uplus \{x = t\}$ and $x \not\in \text{vars}(t)$ and $x \not\in \text{vars}(E')$.

Standard (naive) Unification

Tautology $E \uplus \{t = t\} \Rightarrow_{SU} E$

Decomposition $E \uplus \{f(s_1, \dots, s_n) = f(t_1, \dots, t_n)\} \Rightarrow_{SU}$
$E \cup \{s_1 = t_1, \dots, s_n = t_n\}$

Clash $E \uplus \{f(s_1, \dots, s_n) = g(s_1, \dots, s_m)\} \Rightarrow_{SU} \bot$
if $f \neq g$

Substitution $E \uplus \{x = t\} \Rightarrow_{\mathrm{SU}} E\{x \mapsto t\} \cup \{x = t\}$

if $x \in \mathit{vars}(E)$ and $x \notin \mathit{vars}(t)$

Occurs Check $E \uplus \{x = t\} \Rightarrow_{\mathrm{SU}} \bot$

if $x \neq t$ and $x \in \mathit{vars}(t)$

Orient $E \uplus \{t = x\} \Rightarrow_{\mathrm{SU}} E \cup \{x = t\}$

if $t \notin \mathcal{X}$

3.7.2 Theorem (Soundness, Completeness and Termination of $\Rightarrow_{\text{SU}}$)

If s, t are two terms with $\text{sort}(s) = \text{sort}(t)$ then

1. if $\{s = t\} \Rightarrow^*_{\text{SU}} E$ then any equation $(s' = t') \in E$ is well-sorted, i.e., $\text{sort}(s') = \text{sort}(t')$.
2. $\Rightarrow_{\text{SU}}$ terminates on $\{s = t\}$.
3. if $\{s = t\} \Rightarrow^*_{\text{SU}} E$ then σ is a unifier (mgu) of E iff σ is a unifier (mgu) of $\{s = t\}$.
4. if $\{s = t\} \Rightarrow^*_{\text{SU}} \bot$ then s and t are not unifiable.
5. if $\{s = t\} \Rightarrow^*_{\text{SU}} \{x_1 = t_1, \ldots, x_n = t_n\}$ and this is a normal form, then $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ is an mgu of s, t.