Orderings

Propositional superposition is based on an ordering on the propositional variables, Section 2.7. The ordering is total and well-founded. Basically, propositional variables correspond to ground atoms in first-order logic.

This section generalizes the ideas of the propositional superposition ordering to first-order logic. In first-order logic the ordering has to also consider terms and variables and operations on terms like the application of a substitution. See the first-order resolution calculus.

I first define the ordering on terms and then explain how it is extended to atoms.

3.11.1 Definition (Σ-Operation Compatible Relation)

A binary relation $\sqsupset$ over $T(\Sigma, \mathcal{X})$ is called *compatible with Σ-operations,* if $s \sqsupset s'$ implies
$$f(t_1, \ldots, s, \ldots, t_n) \sqsupset f(t_1, \ldots, s', \ldots, t_n)$$
for all $f \in \Omega$ and $s, s', t_i \in T(\Sigma, \mathcal{X})$.

3.11.2 Lemma (Σ-Operation Compatible Relation)

A relation $\sqsupset$ is compatible with Σ-operations iff $s \sqsupset s'$ implies $t[s]_p \sqsupset t[s']_p$ for all $s, s', t \in T(\Sigma, \mathcal{X})$ and $p \in pos(t)$.

3.11.3 Definition (Substitution Stable Relation, Rewrite Relation)

A binary relation $\sqsupset$ over $T(\Sigma, \mathcal{X})$ is called *stable under substitutions*, if $s \sqsupset s'$ implies $s\sigma \sqsupset s'\sigma$ for all $s, s' \in T(\Sigma, \mathcal{X})$ and substitutions σ.

A binary relation $\sqsupset$ is called a *rewrite relation*, if it is compatible with Σ-operations and stable under substitutions. A *rewrite ordering* is then an ordering that is a rewrite relation.

3.11.4 Definition (Subterm Ordering)

The *proper subterm ordering* $s > t$ is defined by $s > t$ iff $s|_p = t$ for some position $p \neq \epsilon$ of *s*.

3.11.5 Definition (Simplification Ordering)

A rewrite ordering $\succ$ over $T(\Sigma, \mathcal{X})$ is called *simplification ordering*, if it enjoys the *subterm property* $s \succ t$ implies $s > t$ for all $s, t \in T(\Sigma, \mathcal{X})$ of the same sort.

3.11.6 Definition (Lexicographical Path Ordering (LPO))

Let $\Sigma = (\mathcal{S}, \Omega, \Pi)$ be a signature and let $\succ$ be a strict partial ordering on operator symbols in Ω, called *precedence*. The *lexicographical path ordering* $\succ_{lpo}$ on $T(\Sigma, \mathcal{X})$ is defined as follows: if s, t are terms in $T_S(\Sigma, \mathcal{X})$ then $s \succ_{lpo} t$ iff

1. $t = x \in \mathcal{X}$, $x \in vars(s)$ and $s \neq t$ or
2. $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$ and
   a) $s_i \succeq_{lpo} t$ for some $i \in \{1, \ldots, n\}$ or
   b) $f \succ g$ and $s \succ_{lpo} t_j$ for every $j \in \{1, \ldots, m\}$ or
   c) $f = g$, $s \succ_{lpo} t_j$ for every $j \in \{1, \ldots, m\}$ and
      $(s_1, \ldots, s_n)(\succ_{lpo})_{lex}(t_1, \ldots, t_m)$.

3.11.7 Theorem (LPO Properties)

1. The LPO is a rewrite ordering.
2. LPO enjoys the subterm property, hence is a simplification ordering.
3. If the precedence $\succ$ is total on Ω then $\succ_{lpo}$ is total on the set of ground terms $T(\Sigma)$.
4. If Ω is finite then $\succ_{lpo}$ is well-founded.

3.11.9 Definition (The Knuth-Bendix Ordering)

Let $\Sigma = (\mathcal{S}, \Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω, let $w : \Omega \cup \mathcal{X} \to \mathbb{R}^+$ be a *weight function*, so that the following condition is satisfied:
$w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in \mathcal{X}$; $w(c) \geq w_0$ for all constants $c \in \Omega$.

Then, the weight function w can be extended to terms recursively:

$$w(f(t_1, \ldots, t_n)) = w(f) + \sum_{1 \leq i \leq n} w(t_i)$$

3.11.9 Definition (The Knuth-Bendix Ordering Ctd.)

or alternatively

$$\sum w(t) = \sum_{x \in vars(t)} w(x) \cdot \#(x, t) + \sum_{f \in \Omega} w(f) \cdot \#(f, t)$$

where $\#(a, t)$ is the number of occurrences of a in t.
The *Knuth-Bendix ordering* $\succ_{kbo}$ on $T(\Sigma, \mathcal{X})$ induced by $\succ$ and admissible w is defined by: $s \succ_{kbo} t$ iff

1. $\#(x, s) \geq \#(x, t)$ for all variables x and $w(s) > w(t)$, or
2. $\#(x, s) \geq \#(x, t)$ for all variables x, $w(s) = w(t)$, and
   (a) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f \succ g$, or
   (b) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and
       $(s_1, \ldots, s_m)(\succ_{kbo})_{lex}(t_1, \ldots, t_m)$.

3.11.10 Theorem (KBO Properties)

1. The KBO is a rewrite ordering.
2. KBO enjoys the subterm property, hence is a simplification ordering.
3. If the precedence $\succ$ is total on Ω then $\succ_{kbo}$ is total on the set of ground terms $T(\Sigma)$.
4. If Ω is finite then $\succ_{kbo}$ is well-founded.

The LPO ordering as well as the KBO ordering can be extended to atoms in a straightforward way. The precedence $\succ$ is extended to Π. For LPO atoms are then compared according to Definition 3.11.6-2. For KBO the weight function w is also extended to atoms by giving predicates a non-zero positive weight and then atoms are compared according to terms.

Actually, since atoms are never substituted for variables in first-order logic, an alternative to the above would be to first compare the predicate symbols and let $\succ$ decide the ordering. Only if the atoms share the same predicate symbol, the argument terms are considered, e.g., in a lexicographic way and are then compared with respect to KBO or LPO, respectively.