Propositional Tableau

2.4.1 Definition (α-, β-Formulas)

A formula ϕ is called an α-formula if ϕ is a formula $\neg\neg\phi_1$, $\phi_1 \wedge \phi_2$, $\phi_1 \leftrightarrow \phi_2$, $\neg(\phi_1 \vee \phi_2)$, or $\neg(\phi_1 \rightarrow \phi_2)$.

A formula ϕ is called a β-formula if ϕ is a formula $\phi_1 \vee \phi_2$, $\phi_1 \rightarrow \phi_2$, $\neg(\phi_1 \wedge \phi_2)$, or $\neg(\phi_1 \leftrightarrow \phi_2)$.

2.4.2 Definition (Direct Descendant)

Given an α- or β-formula ϕ, its direct descendants are as follows:

α	Left Descendant	Right Descendant
$\neg\neg\phi$	ϕ	ϕ
$\phi_1 \wedge \phi_2$	ϕ_1	ϕ_2
$\phi_1 \leftrightarrow \phi_2$	$\phi_1 \rightarrow \phi_2$	$\phi_2 \rightarrow \phi_1$
$\neg(\phi_1 \vee \phi_2)$	$\neg\phi_1$	$\neg\phi_2$
$\neg(\phi_1 \rightarrow \phi_2)$	ϕ_1	$\neg\phi_2$

β	Left Descendant	Right Descendant
$\phi_1 \vee \phi_2$	ϕ_1	ϕ_2
$\phi_1 \rightarrow \phi_2$	$\neg\phi_1$	ϕ_2
$\neg(\phi_1 \wedge \phi_2)$	$\neg\phi_1$	$\neg\phi_2$
$\neg(\phi_1 \leftrightarrow \phi_2)$	$\neg(\phi_1 \rightarrow \phi_2)$	$\neg(\phi_2 \rightarrow \phi_1)$

2.4.3 Proposition ()

For any valuation $\mathcal{A}$:

(i) if ϕ is an α-formula then $\mathcal{A}(\phi) = 1$ iff $\mathcal{A}(\phi_1) = 1$ and $\mathcal{A}(\phi_2) = 1$ for its descendants ϕ_1, ϕ_2.

(ii) if ϕ is a β-formula then $\mathcal{A}(\phi) = 1$ iff $\mathcal{A}(\phi_1) = 1$ or $\mathcal{A}(\phi_2) = 1$ for its descendants ϕ_1, ϕ_2.

Tableau Rewrite System

The tableau calculus operates on states that are sets of sequences of formulas. Semantically, the set represents a disjunction of sequences that are interpreted as conjunctions of the respective formulas.

A sequence of formulas $(\phi_1, \ldots, \phi_n)$ is called *closed* if there are two formulas ϕ_i and ϕ_j in the sequence where $\phi_i = \text{comp}(\phi_j)$.

A state is *closed* if all its formula sequences are closed.

The tableau calculus is a calculus showing unsatisfiability of a formula. Such calculi are called *refutational* calculi. Recall a formula ϕ is valid iff $\neg\phi$ is unsatisfiable.

A formula ϕ occurring in some sequence is called *open* if in case ϕ is an α-formula not both direct descendants are already part of the sequence and if it is a β-formula none of its descendants is part of the sequence.

Tableau Rewrite Rules

α-Expansion $N \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n)\} \Rightarrow_T$
$N \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n, \psi_1, \psi_2)\}$
provided ψ is an open α-formula, ψ_1, ψ_2 its direct descendants and the sequence is not closed.

β-Expansion $N \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n)\} \Rightarrow_T$
$N \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n, \psi_1)\} \uplus \{(\phi_1, \ldots, \psi, \ldots, \phi_n, \psi_2)\}$
provided ψ is an open β-formula, ψ_1, ψ_2 its direct descendants and the sequence is not closed.

Tableau Properties

2.4.4 Theorem (Propositional Tableau is Sound)

If for a formula ϕ the tableau calculus computes $\{(\neg\phi)\} \Rightarrow_{\mathrm{T}}^{*} N$ and N is closed, then ϕ is valid.

2.4.5 Theorem (Propositional Tableau Terminates)

Starting from a start state $\{(\phi)\}$ for some formula ϕ, the relation $\Rightarrow_{\mathrm{T}}^{+}$ is well-founded.

2.4.6 Theorem (Propositional Tableau is Complete)

If ϕ is valid, tableau computes a closed state out of $\{(\neg\phi)\}$.

2.4.7 Corollary (Propositional Tableau generates Models)

Let ϕ be a formula, $\{(\phi)\} \Rightarrow^*_{\mathrm{T}} N$ and $s \in N$ be a sequence that is not closed and neither α-expansion nor β-expansion are applicable to s. Then the literals in s form a (partial) valuation that is a model for ϕ.